This paper introduces a geometric analysis of public goods very similar to that of bilateral monopoly, demonstrating the essential identity of the two theories. Various public-finance theories are then illustrated: Their basic but often vaguely outlined ideas are elucidated, and implicit assumptions are unveiled in order to show that their different implicit behavioral assumptions are largely responsible for their disagreements. It is found that, under the usual behavioral assumptions, a unique optimum solution cannot be attained spontaneously, but a unique set of possible solutions can be delimited. The importance of income-redistribution effects, hitherto neglected by theorists examining the bargaining solution for an optimum budget, is also demonstrated.

Since the turn of the century, public-finance theorists have been aware that the determination of public expenditure and the distribution of the tax bill under a democratic government involve a process akin to that of economic bargaining not governed by competition, namely, bargaining between a few self-interested parties contracting in the absence of external pressure.¹ Yet the findings of bilateral monopoly theories, which have grown up earlier, have been slow to be integrated into the theory of public finance.²

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¹ For example, Edgeworth (1897; henceforth my references to reprint) and Lindahl (1919; henceforth my references to English translation). A detailed account of the earlier works of public finance can be found in Musgrave (1939).

² Some of the recent studies in public finance, particularly Johansen (1963), Head (1964), and Campa (1967), have however, specifically drawn an analogy between them.
This paper attempts to demonstrate, first, that the pure theory of public expenditure formalized in terms of a collective-consumption (or public) good can in fact be shown diagrammatically to be very similar to a case of bilateral monopoly; and that, when this is done, we can illustrate at once that practically all the important findings of the bilateral monopoly theorists are equally applicable to the pure theory of public expenditure; and, therefore, many problems once thought of as peculiar to the pure theory of public expenditure turn out to be versions of the familiar themes that economists have long been accustomed to identify with the bilateral monopoly case. The diagrammatic exposition is relevant to both public goods and goods imposing externalities. The first section of the paper, therefore, includes a brief comparison of these two cases, arguing that the differences between them lie chiefly in their implicitly assumed institutional and/or technological conditions affecting bargaining possibilities between the parties involved.

Second, the paper relates our diagrammatic exposition of the determination of public expenditure to earlier studies in this area in order to elucidate as clearly as possible a number of points that have been obscured by others and to allow us to appraise their analyses in detail. It will set out the basic ideas commonly but often vaguely conceived by various public finance theorists and show that a number of disputes taking place among them about the possibilities of securing a unique equilibrium solution by a voluntary measure are closely parallel to those that took place among the bilateral monopoly theorists and are often traceable, as in the bilateral monopoly case, to differences in the behavioral as well as institutional assumption that the debaters implicitly or explicitly care to make.

Third, the pure theory of public finance, as in bilateral monopoly theory, will yield multiple solutions unless other "noneconomic variables" are introduced. Accordingly, the paper endeavors to delimit the range of possible equilibrium solutions or the "trading area" in terms of combinations of the public- and private-product mix and the tax bill distribution for various cases, including cases for increasing relative marginal costs of public expenditure. The exercise clearly demonstrates the importance of the hitherto-neglected income-redistribution effect which almost invariably accompanies any sizable public-good production. The paper will be concerned throughout with the question of under what conditions we can (or cannot) rely upon bargaining between the parties involved as a voluntary measure of securing a Pareto-preferred solution and a Pareto-optimum (or an efficient) solution and then of obtaining the social welfare maximum solution.
I. The Model and Its Diagrammatic Representation

Let us consider a social group consisting of two parties, party $A$ and party $B$, each of which consists of many individuals but has a consistent set of ordinal preferences with respect to various goods (and services), which can be represented by a regularly smooth, convex utility function given as, respectively,

\[ U^A = U^A(X^A_i, X_n), \]
\[ U^B = U^B(X^B_i, X_n), \]

\[(i = 1, 2, \ldots, j, \ldots, n - 1), \]

where $X^A_i$ represents party $A$'s share of good $i$. Good $j$ is the numeraire. Equations (1) and (2) establish a consumption interdependency, for good $n$ enters simultaneously into the utility functions of both parties. Following Samuelson (1954), good $n$ can be defined as a collective-consumption good or a public good, for $X_n = X^A_n = X^B_n$, while $X^A_i$'s are private goods and $X^B_i = X^A_i + X^B_n$. Using the convention of writing the partial derivative of any function with respect to its $i$th argument by an $i$ subscript so that

\[ u^A_{x_i} = \frac{\partial U^A}{\partial X^A_i}, \]

and so forth, and by properly arranging the algebraic signs of all goods, we can assume that

\[ u^k_{x_i} > 0 \quad (k = A, B) \quad (i = 1, \ldots, j, \ldots, n - 1), \]

and

\[ u^B_{x_n} > 0, \quad \text{but} \quad u^A_{x_n} \neq 0. \]

The transformation function relating all outputs and inputs is assumed to display the characteristics of constant returns to scale and smooth and continuous convex isoquants and is given as

\[ f(X_1, X_2, \ldots, X_j, \ldots, X_n) = 0, \]

with

\[ fX_i > 0 \quad (i = 1, 2, \ldots, j, \ldots, n). \]

We assume that all factors of production are inelastically supplied and privately owned by individuals and that there exists competition in production.

Diagrammatically, the functional relationships shown in equations (1), (2), and (3) can be represented as figures 1, 2, and 3, respectively. As shown in figures 1 and 2, we shall be explicitly concerned throughout with the case where both parties regard $X_n$ as a "good" (that is, the reader is asked to adjust the analysis himself should he be interested
in the case where one party regards $X_n$ as a "good" and the other party as a "bad"); and, as figure 3 shows, we shall assume a linear transformation line first, deferring discussions on the case where the transforma-

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For example, if party $A$ regards $X_n$ as a bad (that is, $u^A_n < 0$), $A$'s indifference curves have positive slopes. The analysis remains, nevertheless, substantially the same.
tion curve is nonlinear until Section III. Each diagram has the numeraire good, $X_j$, on its vertical axis and $A_o D_a$ of figure 1 and $B_o D_b$ of figure 2 indicate, respectively, $A$’s and $B$’s generalized purchasing powers or “incomes” in terms of the numeraire good. The sum of $A_o D_a$ and $B_o D_b$ makes up the total income of the two parties combined and equals $AB$ in figure 3. Since $X_n = X_n^A = X_n^B$, each diagram is lined up with exactly the same horizontal scale. The three diagrams are thus interrelated, and obviously the interrelationships of the three will be much clearer if they are integrated into one diagram. This is done in the following procedure:

First, we rotate the horizontal axis ($X_n$ axis) of figure 2 around its origin, $B_o$, so that the rotated axis, denoted as $B_o X^*_n$, has the same slope, in absolute terms, as the production-possibility curve of figure 3 at each output level of $X_n$ with respect to the $B_o X_n$ axis. We then redraw each indifference contour for $B$ as a broken line by measuring the quantities of $X_j$ and $X_n$ required for $B$ to remain at the same utility index by the vertical distance from the rotated axis ($B_o X^*_n$) and by the horizontal distance from the vertical axis ($B_o X_j$), respectively. This operation produces a new set of broken-line contours, which are now called adjusted indifference curves for $B$, as shown in figure 2. Clearly, the map of the adjusted indifference curves conveys exactly the same information as the original map of indifference curves, the only dif-
ference being that the slope of an adjusted indifference curve at any point with respect to $B_o X_n$ axis is now

$$(-)u^B_{t_n}/u^B_{t_j} + f_{x_n}/f_{x_j}.$$  

The second step is that we flip up and over the map of $B$’s adjusted indifference curves around the horizontal line crossing point $D_b$, or “money” income of $B$, $D_b F$, by 180 degrees and fit it on the top of the $A$’s indifference map so that the $A_o X_j$ axis of figure 1 and the $B_o X_j$ axis of figure 2 become the same axis ($A_o X_j$) and point $D_b$ coincides with point $D_a$, which indicates “money” income of $A$. This simple two-step transformation produces a right-angle triangle, as shown in figure 4, congruent to the production-possibility block of figure 3. It is immediately apparent that the triangle diagram thus produced in figure 4 not only shows the production-possibility relationship but also possesses essentially the same characteristics as those of the familiar Edgeworth-Bowley box diagram. In fact, it is so similar in nature that a brief explanation will perhaps suffice for our needs.

In figure 4, $AB$ on the vertical axis shows the endowment of the group as a whole measured, of course, in terms of the numeraire commodity or the sum of the initial “incomes” of parties $A$ and $B$, with $D$ being the distribution point of the sum between them. While $BC$ is the group’s production-possibility curve, the horizontal line $DF$ is party $B$’s production-possibility curve when $B$’s initial endowment alone is available for production of $X_n$. Therefore, $DF$ may be called party $B$’s independently attainable consumption-possibility frontier. Similarly, party $A$’s production-possibility curve, and hence its independently attainable consumption-possibility frontier, is drawn as a line parallel to $BC$ from $D$ toward $A X_n$ axis as shown by line $DE$. Any combination of public and private goods that falls between these two frontiers is not obtainable by either parity independently. Both parties must jointly finance the costs of $X_n$. For example, to reach a combination of $X_n$, $X^B_j$, and $X^A_j$, represented by point $J$, $A$ and $B$ must bear taxes corresponding to $JK$ and $LJ$ of the numeraire good, respectively. Their respective indifference curves containing point $J$ indicate their welfare levels thus attainable jointly.

Figure 4 was developed independently in the context of a public good. However, it came to my attention later that in the context of external diseconomy Dolbear (1967) reached a similar diagram by a somewhat different method. But, in my view, Dolbear’s “fold-in” and “stretch” method is not entirely without ambiguity, particularly when the transformation line is nonlinear, whereas the method presented here has the advantage of being applicable to any forms of transformation curves without ambiguity, as will be seen in Section III.
The slope of an adjusted indifference curve (of B) with respect to A Xn axis is now
\[
\frac{u_{x_n}^B}{u_{x_j}^B} - f_{x_n}/f_{x_j}.
\] (4)

Therefore, at any tangency point between A’s indifference curve and B’s adjusted indifference curve, the following relationship holds:
\[
(-)\frac{u_{x_n}^A}{u_{x_j}^A} = \frac{u_{x_n}^B}{u_{x_j}^B} - f_{x_n}/f_{x_j},
\] (5)
or
\[
\frac{u_{x_n}^A}{u_{x_j}^A} + \frac{u_{x_n}^B}{u_{x_j}^B} = f_{x_n}/f_{x_j}.
\] (6)

Equation (6) is the well-known Pareto-optimum condition for a public good specified by Samuelson (1954) in his theory of public expenditure, and equation (5) is the definition of Pareto equilibrium used by Buchanan and Stubblebine (1962) in their paper on externality. Thus the locus of tangency points between A’s indifference curves and B’s adjusted indifference curves, VW, represents a set of Pareto-optimum (efficient) configurations of values of X4, X4, and Xn, or a “contract curve.” Tracing A’s and B’s welfare indexes along the contract curve, we can generate the Samuelsonian utility-possibility frontier curve of this social group (Samuelson 1955, p. 352, chart 4). The contract curve shows at
once a number of points stressed by Samuelson on several occasions (1954, 1955, 1958): (1) There are multiples of Pareto-optimum solutions; (2) to select out of these multisolutions a single solution as the best for the group as a whole, a social welfare function of Bergson type, that is, \( U = (U^A, U^B) \), that provides an ethical evaluation of the well-beings of different parties is required; and (3) so long as the contract curve does not happen to be a vertical straight line, the economic-efficiency problem (namely, how much of a public good it is efficient to produce) and the ethical problem (namely, how in justice the costs of public expenditure are to be distributed between the parties) cannot be logically separated; indeed, as is clearly shown in figure 4, choosing a point on the contract curve amounts to resolving these two problems simultaneously.\(^5\)

It is clear from figure 4 that wherever the initial position happens to be at that position, if the indifference curve of \( A \) and the adjusted indifference curve of \( B \) are not tangent to each other, or

\[
(-) \frac{u^A_{z_i}}{u^B_{z_i}} \neq \left( \frac{u^B_{z_i}}{u^A_{z_i}} - \frac{f^B_{z_i}}{f^A_{z_i}} \right),
\]

there generally exists a set of other solutions which are preferable to that initial position in the sense that, by moving to one of these solutions, party \( A \) (\( B \)) can be made better off without party \( B \) (\( A \)) being made worse off. Such a set of solutions, which shall hereafter be called “Pareto-preferred solutions” of that initial position, is represented by an area demarcated by the indifference curve of \( A \) and the adjusted indifference curve of \( B \), both of which contain that initial position (that is, DIY and DHZ if the initial position is \( D \)). If \( A \) and \( B \) are able to trade and the trade is in fact carried out, they will move on to a Pareto-preferred solution at each successive stage of trade; and an opportunity of trade ultimately disappears when they reach a solution on the contract curve, namely, a Pareto-optimum solution. Thus any set of Pareto-preferred solutions normally contains a subset of Pareto-optimum solutions, which shall be henceforth referred to as a set of “Pareto-preferred Pareto optima.” As opposed to a Pareto optimum, a Pareto-preferred Pareto optimum requires a reference point for its identification. The same Pareto-optimum solution could be either Pareto preferred or non-Pareto preferred, depending upon the location of the reference point from which preferability of the new solution in question is examined.

\(^5\) Figure 4 clearly shows that if the marginal utility of “income” is constant for both parties, the contract curve becomes a vertical straight line, and, consequently, the determination of the optimum quantity of \( X \) (that is, the allocation problem) is independent of the distribution of welfare between the parties (that is, the distribution problem). This result is often stated or implied without due consideration or explicit recognition of the above condition needed to uphold it. (For a different diagrammatic presentation of the discussion above, see McGuire and Aaron [1969].)
Undoubtedly, recognition of the existence of a solution that is Pareto preferred to the initial position, which is usually in a discussion of public goods a position such as $D$, where the public good in question is absent, must be the basis of the so-called voluntary-exchange theory of public finance. Lindahl (1919) observed, for example, that "the sum of the contributions which the various parties may be prepared to make towards the realization of the more important collective goods far exceeds the latters' total cost" (p. 168). This means to say in our figure 4 that, to realize a quantity—say $AG$—of good $n$, the maximum contributions which party $A$ and party $B$ will be prepared to make if necessary are $IK$ and $LH$, respectively, and their sum far exceeds the total cost necessary to produce that quantity ($AG$) of the public good, that is $LK (=NB)$.

But it is at once clear from our diagram that the situation at hand is essentially one of indeterminate bilateral monopoly. We cannot expect that each party will voluntarily contribute toward the collective-consumption good unless it is otherwise precluded from enjoyment of the public good by some exclusion devices. Each party, in the hope of moving along the opponent's indifference curve constituting the outer edge of the area of Pareto-preferred solutions ($DHZ$ or $DIY$), will most likely engage in a game-theoretic strategy.

Economists customarily have had very little to say about pure bargaining situations in which the outcome is dependent upon interactions among only a few parties. Except for a few models based on strong institutional assumptions, we have had to fall back on Edge-worth's model of bilateral monopoly, in which only a "trading area" is delimited with no further restrictions of the outcome. Within the trading area, the solution is said to depend upon the "bargaining abilities" of the parties. Unsatisfactory though it may be, a democratic determination of public expenditure squarely falls into this category of situations. The most we can do is to identify a set of possible solutions or the trading area. The trading area is delimited in our diagram generally as the area demarcated by two indifference curves, each of which is the highest indifference curve that each party can attain independently from the starting point in question. Such an indifference curve is, as a rule, the one tangent to the respective party's independently attainable consumption-possibility frontier. For example, in figure 4, evidently if $RY'$ ($SZ'$), the indifference curve tangent to $DE$ ($DF$), represents a higher welfare level for party $A$ ($B$) than that party's initial welfare level represented by $DIY$ ($DHZ$), $A$ ($B$) will not accept a solution whose welfare level is lower than $RY'$ ($SZ'$) even though it is a Pareto-preferred solution of the initial position represented by $D$, and the trading area

* An exception to this will be discussed in Section III.
in this instance is the area demarcated by indifference curves $RY'$ and $SZ'$, which is a subset of the Pareto-preferred solutions of the initial position. Thus, the area of Pareto-preferred solutions does not necessarily coincide with the trading area: Indeed as will become clear later, the two concepts are distinctively different from each other.

The trading area thus delimited clearly shows that possible solution points are located on both the left and right sides of the contract curve, and, consequently, contrary to the observations of some economists, we have no logical reason to “observe that all possible [solution] points apart from $P$ [a point on the contract curve which will be specified later in fig. 5] imply a smaller amount of public expenditure than the optimal one.” Indeed, the pure theory of public expenditure itself does not appear to offer any clue to an explanation of the opinion expressed by many observers that public expenditures are suboptimal in the West European and North American countries. This type of observation seems to have stemmed partly from the observers’ failure to reject completely the significance of Lindahl’s demand curves in favor of the recognition of the trading area and partly their neglect of the income-redistribution effect. We shall revert to this problem in the next section in connection with Lindahl’s analysis and again in Section III in connection with the increasing marginal cost case.7

The case of a public good discussed above can be contrasted with the case of a good which imposes an external effect on others. In usual discussions on consumption externality, the following relationships are assumed:8

$$U_A = U_A(X_A, X_B)$$

$$U_B = U_B(X_A, X_B);$$

$$J_B(X_A, X_B) = 0 \quad (i = 1, 2, \ldots, j, \ldots, n - 1);$$

and

$$J_B f_{x_b}/f_{x_j} < u_B^B/ u_j^B \quad \text{when} \quad X_n^B = 0.$$  

Thus the quantity of $X_B^n$, which simultaneously enters utility functions of more than one party (two parties in this case), is assumed to be controlled and produced only by one party (in this case party $B$), called the acting party, on the assumption that it is the controller of the good which imposes an externality on the other. Clearly, the basic characteristic of this externality relationship is identical to that of a public good expressed by equations (1) and (2); and it is not difficult to see that the necessary condition for Pareto optimality for this case is identical to that of a public good shown by equation (5) or equation (6).

7 See n. 26.

8 For example, Buchanan and Stubblebine (1962), where condition (10) is not explicit, however.
If there are any features that distinguish an externality relationship from a public-good case, they are the assumed institutional and/or technological conditions including the structure of property rights that motivate and/or constrain the behavior of the parties involved. In the externality case, for example, it is usually assumed, but often implicitly, that for institutional and/or technological reasons the parties involved cannot or will not enter into effective bargaining regarding the provision of $X_B^n$, and, therefore, the party controlling the good or activity creating the externality (party $B$) reaches its own equilibrium position in isolation. With the acting party, namely, party $B$, having reached its private equilibrium, we shall then have the following situation at that private equilibrium position which is often referred to as the status quo situation:

$$f^B_{x_i}/f^B_{x_j} - u^B_{x_i}/u^B_{x_j} = 0,$$

where $X^B_n = X_n^{BS}$

assuming, of course,

$$f^B_{x_i} > 0 \quad \text{and} \quad u^B_{x_i} > 0 \quad (i = 1, 2, \ldots, j, \ldots, n).$$

It is then defined that a marginal externality exists when

$$u^A_{x_i} > 0, \quad \text{and} \quad u^A_{x_i}/u^A_{x_j} \neq 0 \quad \text{for} \quad X^B_n = X_n^{BS}.$$  

It is a marginal external economy if

$$u^A_{x_i}/u^A_{x_j} > 0 \quad \text{for} \quad X^B_n = X_n^{BS},$$

and is a marginal external diseconomy if

$$u^A_{x_i}/u^A_{x_j} < 0 \quad \text{for} \quad X^B_n = X_n^{BS}.$$  

In figure 4, such a private equilibrium or the status quo position is indicated by point $S$, where one of party $B$'s adjusted indifference curves is tangent to $B$'s independently attainable consumption-possibility frontier, $DF$, (that is, $B$'s production-possibility curve). If $A$'s indifference curve has a negative slope at $S$, $B$'s provision of $X_n$ creates a marginal external economy to $A$. Alternatively, if it has a positive slope, $B$ imposes a marginal external diseconomy on $A$. The Pareto-preferred solutions of this status quo position ($S$) are the area delimited by $A$'s and $B$'s indifference curves containing point $S$. However, as stated, by assumption party $A$ and party $B$ cannot or will not enter into bargaining; with absence of private trade between them, they cannot move to a Pareto-preferred solution of $S$ by themselves and remain at $S$. In other words, in the case of externality, to dislodge the parties from the status quo position and lead them to a Pareto-preferred or a Pareto-optimum
solution, an intervention by the third party (say the government) is required. Many discussions on external economy, however, often casually suggest, explicitly or implicitly, that private trade is possible starting from the private-equilibrium point such as $S$. But, if private trade is possible at all, we have no logical reason why such a point as $S$ ought to be considered as the starting point of negotiation or status quo position, and the situation must be conceived as that of a public good in which a strategic consideration most likely prevents the acting party from reaching such a point as $S$ before entering negotiation.9

II. Relations with Other Theories

When figure 4 is related to various earlier theories of public finance, it reveals a number of points which hitherto have not been brought forward clearly and allows us to appreciate their analyses. Wicksell (1896; henceforth my references to English translation) sought a solution for the quasi-bilateral bargaining situation involved in determination of public expenditure in Parliament or political democracy. He proposed simultaneous voting on expenditure and taxes on the ground that: "Provided the expenditure in question holds out any prospect at all of creating utility exceeding costs, it will always be theoretically possible, and approximately so in practice, to find a distribution of costs such that all parties regard the expenditure as beneficial and may therefore approve it unanimously. Should this prove altogether impossible, I would consider such failure as an a posteriori, and the sole possible, proof that the state activity under consideration would not provide the community with utility corresponding to the necessary sacrifice and should hence be rejected on rational grounds" (p. 89). Wicksell thus apparently based his argument on an optimistic view that so long as a proposed solution was a Pareto-preferred solution, all parties concerned would approve it unanimously. However, the validity of this view is in doubt in the light of the modern game theory. It might be possible to reach a unanimous consent if there were only one Pareto-preferred

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9 If an externality relationship is reciprocal in the sense that

$$U^A = U^A(X^A_i, X^A_n, X^B_n)$$ and

$$U^B = U^B(X^B_i, X^B_n, X^A_n),$$

it involves indeterminacy of the status quo position similar to that of a public good insofar as $X^A_n$ and $X^B_n$ are substitutes. Indeed, in the extreme case, namely, where they are perfect substitutes—that is, each party does not distinguish its own consumption of $X_n$ from the indirect consumption of the other—marginal reciprocal externalities of the separable kind are reduced to public goods. Or alternatively, a public good could be defined as the limiting case of the separable reciprocal externality. For separable and nonseparable externalities, see Davis and Whinston (1962).
solution; but, as figure 4 shows, the number of Pareto-preferred solutions is infinite, and so is that of Pareto-preferred Pareto optima. Each party striving to obtain a solution as close as possible to the opponent's indifference curve constituting the outer edge of the Pareto-preferred solution area will most likely refuse to make the vote on any particular solution unanimous unless the vote is an all-or-none offer, which is clearly not the voting procedure that Wicksell was proposing. The difficulty will be intensified rather than removed as the number of parties involved is increased for, with respect to any proposed solution, almost all parties except the one proposing it would find some alternatives preferable, and be prepared to propose their own alternatives, and try to get them passed by voting down the proposed.\(^{10}\)

In fact, the unanimous-voting approach appears to make the problem of determining a single solution in a situation like a bilateral monopoly even less tractable than the direct-bargaining approach. It is generally argued in the literature on bilateral monopoly that, to discover a possible equilibrium position, the modes of behavior of the parties involved must be known in addition to their objectives, and the equilibrium position will depend upon the "bargaining abilities" of the parties. In this respect, Lindahl (1919), who apparently sensed the indeterminacy of the Wicksellian approach and attempted to solve the problem by introducing the concept of the equal bargaining power as an additional determinant and assuming—albeit implicitly—specific behavioral patterns for the two parties involved, thus appears to have been more in line with the modern approach to the duopoly problem.

Lindahl's solution is a Pareto-optimum point on the contract curve. It is defined in our diagram as the intersection of the locus of points of tangency between party A's indifference curves and various rays extended from point D and the locus of points of tangency between the same rays and party B's adjusted indifference curves. The slope of such a ray indicates a distribution ratio of the tax bill between the two parties, and, of course, the point of tangency between the ray and one of the indifference curves of one party indicates the optimum combination of the public and private goods that that party would choose if it had to

\(^{10}\) The time constraint or impatience, namely, a high opportunity cost involved in remaining at the status quo position, costs of administering voting, and other technical and institutional constraints may make a particular vote effectively an all-or-none offer for all parties concerned. In such a case, a solution located within the trading area would secure unanimity. But if there are no such external pressures and tabling counterproposals is always permitted, as advocated by Wicksell (1896, p. 92), there will be little chance of any specific proposals being accepted by a unanimous vote. Moreover, yielding to the Wicksellian qualified majority is not only often insufficient to secure a decision but also self-defeating for the very idea of the qualified majority must imply "interpersonal utility comparison," which Wicksell was apparently striving to avoid. (Another difficulty of the Wicksellian approach will be discussed in Section III.)
bear the tax share represented by that ray. In figure 5, reproduced from figure 4, the loci of such tangency points (analogous to the lines commonly known as price-consumption curves and to be called hereinafter the pseudo offer curves) are denoted by $D^A$ and $D^B$ for parties $A$ and $B$, respectively, and their intersection point by $P$. Pseudo offer curves $D^A$ and $D^B$ and point $P$ correspond to curves $RA$ and $SB$ and point $P$, respectively, of Lindahl's figure 1 (1919, p. 170). Since at the point of intersection of $D^A$ and $D^B$, both party $A$'s and party $B$'s (adjusted) indifference curves are tangent to the same ray, they must have the same slope. Therefore, point $P$, the intersection point, must be on the contract curve. Clearly, Lindahl's solution is the equivalent of the perfectly competitive market solution of private goods, which can similarly be represented as the point of intersection of the two offer curves in an Edgeworth-Bowley Box Diagram.

It is well known, however, that the offer curves will be revealed only when each party acts as a price taker or considers that its own price, that is, tax-share ratio in this case, is uncontrollable by its own action. However, different from the perfectly competitive market of private

![Fig. 5](image-url)

Johansen (1963) made a conceptually similar but diagrammatically different approach to derive Lindahl's $RA$ and $SB$ curves.
goods, in the case of a public good each party is perfectly well aware of its influence on its own price. The offer curves for a public good will thus never be revealed. (Therefore, the names of pseudo offer curves.) But they perform a useful function to elucidate the similarities and differences in the market mechanism between the public and private goods.

Figure 5 shows that, should the pseudo offer curves have ever been revealed or should all taxpayers have acted as the price taker, a central coordinating agency would find by, say, a process analogous to the Walrasian tâtonnement, an equilibrium quantity of $X_n$, represented by such a point as $P$. Point $P$ may be called, therefore, the pseudo laissez faire solution corresponding to the given distributions of ownership of factors of production, implicit in the income-distribution pattern indicated by point $D$. All that the servant of the ethical observer had to do to swing the economy to his master's grand social optimum would be, exactly as in the case of a competitive private-goods market, to shift the starting point of the pseudo offer curves, $D$, along the vertical axis by varying the initial distribution of generalized purchasing power, namely, the initial $X_i^4$ and $X_i^p$, by a lump-sum tax and transfer, that is, Samuelson's $L^t$ (1954, p. 388), until the intersection point of the pseudo offer curves would land on the attainable-bliss point on the contract curve. The conception of this pseudo competitive market solution seems to be the genesis of the Wicksellian separation of the government function into two parts: one to establish a proper (or just) initial state of income (or property) distribution and the other to determine efficient size of public expenditure and allocation of its tax bill through the voluntary contribution. Obviously, however, the whole rationale of the separation and the voluntary-solution arguments will collapse if the pseudo laissez faire solution is unattainable spontaneously.

Lindahl understood well that to trace an equilibrating process of his model he cannot use an assumption that both parties act as price takers or quantity adjusters as would be the case under the perfect competition. His description of the adjustment process clearly indicates (albeit

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12 In the case of the private goods only, in order to swing the economy to the attainable-bliss point on the contract curve, the same operation must be carried out in an Edgeworth-Bowley Box Diagram, the only difference being that the offer curves of this case are actual and not pseudo. Thus, with respect to the need for redistribution to attain the social optimum, there are no differences between the market of private goods only and that involving public goods in spite of Musgrave's (1959, p. 58) statement seemingly implying the opposite.

13 However, there is no guarantee, as is clear from our figure 5 and pointed out by Myrdal (1930; henceforth my references to English translation), that if the income distribution before tax (represented by $D$) is equitable, the postfiscal action welfare distribution voluntarily achieved by the pseudo competitive market mechanism (represented by $P$) will be also equitable. In fact, to obtain the equitable postfiscal action distribution, the income distribution before tax per se may more often than not have to be made "inequitable."
implicitly) that he has employed the so-called leadership and follower-ship approach usually attributed to von Stackelberg, namely, one party acts as a price maker and the other as a quantity adjuster. Under this assumption, the price maker fixes the price so that his welfare is maximized, taking into account the fact that the other party will only select points lying on its offer curve. Then the tax-bill distribution ratio that the price maker, say, party A, chooses will be that corresponding to the point at which his indifference curve is tangential to the other party’s pseudo-offer curve (DSb). This point is given by Q in figure 5 and corresponds to point Q of Lindahl’s diagram. It is the Cournot point in the usual theory of monopoly. Another Cournot point, R, can, of course, be obtained by reversing the roles of party A and party B. Point R of figure 5 corresponds to point R of Lindahl’s figure 1, and, as in Lindahl, figure 5 is so drawn that at R party B happens to bear no tax at all.14

Once points Q, the most favorable position for party A, and R, the most favorable position for party B, are identified, it is not too difficult to discover that “each party naturally tries to shift the equilibrium, within that interval, to its own advantage” (Lindahl, 1919 p. 171). Thus anticipating the analysis of von Stackelberg, which was to appear some considerable years later, of a case where both parties were striving for “leadership,” Lindahl had correctly recognized the indeterminacy of the consequences of such a model and simply delimited a range of possible solutions, although incorrectly, to a set of points lying on the curve QPR.

Although Lindahl recognized the model’s indeterminacy, he did not go far enough toward recognizing that once the hypothesis party A makes about party B’s behavior will be proved wrong as soon as it is tested, there is no reason why party A should continue to regard party B’s pseudo-offer curve as party B’s “reaction curve,” and, of course, vice versa. There seems to be no logical reason to believe that the possible equilibrium positions are to be found only on the pseudo-offer curves of the two protagonists (QPR).15

Nevertheless, within the framework of his analysis, where party A’s and party B’s pseudo offer curves are regarded as their only reaction curves, the sole point that satisfies the Pareto-optimum condition is point P. Lindahl apparently regarded point P as the ideal solution from his own normative standpoint, knowing that it was in his system the

14 Incidentally, Lindahl’s points T, a provisional equilibrium point, and U correspond, respectively, in our figure 5 to point T, where the ray representing bisection of the cost intersects party B’s pseudo offer curve, and U, where the same ray is tangent to one of party A’s indifference curves. Lindahl’s point S corresponds also to S in figure 5, where party A’s percentage share of total cost is zero.

15 The same point was made by Johansen (1963, p. 353) and implied by Heal (1964, p. 445).
only efficient solution and that it was comparable to the solution of a perfectly competitive private-goods market; and he apparently wanted to conjure a situation which would lead his system to this particular solution. This was perhaps his undoing. To break the indeterminacy of his model, he introduced the notion of equal bargaining power as an additional determinant to his system. This was sought as an improvement of his approach over the Wicksellian approach, but instead it sent his argument into a circular reasoning. He defined, in effect, the equality in bargaining power as the power relationship that would lead the parties concerned to this particular equilibrium position represented by point \( P \). Clearly this was tautological. True, there might be a certain combination of bargaining power of the parties which precisely leads them to the solution at \( P \), but, unless the bargaining power is defined independently of that consequence and measured somehow in an objective way, the equal bargaining power is an empty concept. To be sure, bargaining between protagonists with equal bargaining power measured in terms of some objective scales, assumed to exist, will lead the parties on to the solution represented by some point in the trading area, but it will be an accident if such a point coincides with point \( P \). Nor is there any assurance that his assumption of "just initial distribution of property" necessarily leads to this solution.\(^{16}\) Lindahl's rigid adherence to the behavioral assumption that each party reacts always along its pseudo demand curve and his willingness to dodge the real issue by the tautology of the equal bargaining power seem to be the main weaknesses of his otherwise pioneering work.

Musgrave (1959), criticizing Lindahl's analysis of the equilibrating process, argued that in "the case of two tax-payers only, we must have a solution analogous to the Cournot view of duopoly pricing" and, therefore, "following the Cournot case, we suppose that \( A \) and \( B \) both disregard the effect of their votes upon the other's cost share." On this model he concludes that, "given the assumption that both disregard the effects of their bidding on price, there is nothing in the mechanism of adjustment that makes for a change in cost shares." He indicates in his figure 4-2 that, if an arbitrary cost-distribution ratio is given initially, the equilibrium output of a public good will be established at the lower of the two levels of output, one of which is desired by one party and the other by the other at that given price or cost-distribution ratio (Musgrave 1959, p. 79).

\(^{16}\) In his penetrating review of Lindahl's theory, Head (1964) seemed to have embraced this tautology of Lindahl rather uncritically when he declared, "The crucial role of Lindahl's assumption of 'equal power' in the very special and precise sense of 'ability to exchange up to saturation, given the budget restraint of an initial endowment of factors' should now be quite clear; without it, there can be no guarantee that the fiscal optimum will in fact be achieved and equilibrium established at \( P \)" (p. 447).
Two points may be noted. First, this interpretation by Musgrave of Lindahl’s model seems to be a little less sophisticated than the model that I think Lindahl was actually describing in his analysis. Lindahl’s own description of the equilibrating process appears to be more in line with von Stackelberg’s “leadership and followership” approach, as already discussed above in detail. Second, even if Musgrave’s interpretation is correct, his analysis of the process leading to a unique output “agreed upon by both A and B” is not entirely convincing. Our contention can be illustrated in figure 5. Let a ray extended from D passing through U be assumed to represent the arbitrary rate of cost shares set initially—or $NJ/TJ$ in Musgrave’s figure 4-2. ¹⁷ His arbitrary starting point $J$ is then given as a point located anywhere between $D$ and $T$ on the ray, denoted also as $J$ in our figure 5. Clearly, when the tax-bill share ratio is as represented by the ray $DU$, party B (Musgrave’s A) will vote to expand the public-good output ($X_n$) to the level that corresponds to point $T$ (Musgrave’s $OQ$), where one of its indifference curves is tangent to the tax-share ray (the pseudo offer curve $DSb$ intersects ray $DU$ at $T$) and party A (Musgrave’s B) will vote for the level represented by $U$ (Musgrave’s $OC$), where one of party A’s indifference curves is tangent to the same ray (the pseudo offer curve $DRA$ intersects ray $MU$ at $U$).

Musgrave then proceeds to argue that at the output represented by $U$, party B will be prepared to contribute only its share represented by the ray $DM$, $M$ being the point where the vertical line containing $U$ intersects B’s pseudo offer curve $DSb$. But at this tax-share ratio, party A will not agree to the output of $X_n$ indicated by $U$ (since party A’s pseudo offer curve intersects ray $DM$ at the output much smaller than $U$). Musgrave then argues that party A “will vote for a smaller supply, and adjustment continues until output $OQ$ [represented by $T$ in our figure 5 and also by $T$ in Lindahl’s fig. 1] is reached and agreed upon by both A [party B] and B [party A]. This is not Lindahl’s optimum output. It is the most favorable position in view of the fact that cost shares have been initially set at $NJ/TJ$” (italics mine) (1959, p. 79). The reasoning behind the statement that output $OQ$ “is the most favorable position” is not clear, but it is unmistakably clear from our diagram that, when the tax share is set rigid at the rate represented by ray $DU$,

¹⁷ Note that diagrammatically Musgrave’s taxpayers A and B correspond to Lindahl’s (hence our fig. 5’s) party B and party A, respectively. Therefore, $NJ/NT$, A’s (party B’s) share, and $TJ/NT$, B’s (party A’s) share, correspond to $(1 - \tan \alpha/\tan \gamma)$ and $(\tan \alpha/\tan \gamma)$ in figure 5, respectively, where $\gamma$ is the angle between ray $DE$ and ray $DF$, and $\alpha$ the angle between ray $DU$ and ray $DF$. Consequently, $a_1$ $a_1$ and $b_1$ $b_1$ curves in Musgrave’s figure 4-2 correspond to the pseudo offer curve of party $B$ ($DSb$) and that of party A ($DRA$) in our figure 5, respectively, although there is some ambiguity in Musgrave’s explanation of $a_1$ $a_1$ and $b_1$ $b_1$ curves, as was discussed by Johansen (1963, p. 350).
point $T$ that corresponds to the output $OQ$ of Musgrave’s figure is the most favorable solution for party $B$ but for party $A$ the least favorable of all the possible solutions.\textsuperscript{18} A question then naturally arises: Why should party $A$ have to submit to party $B$ and to agree upon the solution represented by $T$ (Musgrave’s $OQ$)? It might be thought that party $B$ could refuse to contribute any more than the absolute amount of tax needed to produce the output represented by point $T$, and this would leave party $A$ no alternative but to agree with that output.\textsuperscript{19} But this amounts to changing the rules of the game from “the share-ratio rule” to “the absolute-share rule,” and logically frees party $A$ from acting in accordance with the assumed share-ratio rule. Finding that party $B$’s absolute contribution is fixed, party $A$ will then increase its own contribution and expand the output of $X$, up to its own optimum level, namely, a point where its indifference curve is tangent to the line containing point $T$ and parallel to $DE$. This, in turn, will change party $B$’s optimum position and $T$ can no longer be the stable-equilibrium position.\textsuperscript{20} In other words, to maintain consistency of the rules of the game if there are any means in the system of placing one party in a better position than the other, logically the same means must also be made available to the other party. Thus, Musgrave’s analysis seems to have found a unique equilibrium solution only because he either set out with an asymmetric set of the rules of negotiation or implicitly assumed that the bargaining skill of party $B$ was overwhelmingly superior to that of party $A$.

The truth is that the system is at best indeterminate even under the regime of such a restrictive assumption as Musgrave’s that the tax-share ratio is given and unalterable. In figure 5, if the initial starting point is $J$, an equilibrium point could be anywhere on the segment of the ray representing that cost-share ratio, $DU$, between $T$, the most favorable position for party $B$, and $U$, the most favorable position for party $A$. (If, however, party $B$’s indifference curve containing $J$ cuts ray $DU$ between $T$ and $U$, the range of possible equilibrium positions is reduced to a section between $T$ and that cutting point.) Since the most favorable position for one party is the least favorable for the other.
other, each party naturally tries to shift the equilibrium, within that possible range, to its own advantage. Thus the \(TU\) segment of the ray \(DU\) corresponds to the concept of "contract curve." Which out of all the possible solutions will eventually become an equilibrium solution cannot be ascertained unless additional information is given. Thus, even assuming, as does Musgrave, that each party cannot influence its price or tax share by a change in its contribution, a bargaining aspect enters into the determination of public expenditure, and no unique equilibrium solution emerges.

Musgrave correctly realized Lindahl's error of taking two parties' pseudo demand curves to be their only possible reaction curves and hence of delimiting the locus of possible equilibrium positions to a segment of the demand curves, \(QPR\) in Lindahl's figure 1 (or a part of \(UDY\) in Musgrave's figure 4-2). Yet apparently, as Musgrave's analysis shows, he could not shake off completely the same tendency as that of Lindahl of tracing the reactions of parties along their demand curves or of attaching some significance to the pseudo offer curves. Thus, although he correctly expanded the locus of possible equilibrium solutions from a "line" to an "area," he delimited it incorrectly and in an extremely vague way as "certain points on or to the left of \(UDY\." But, as our analysis above shows, the pseudo offer curves are of no significance in delimiting the trading area; possible equilibrium solutions do exist both to the right of and to the left of \(UDY\) (or both above and below \(QPR\) in Lindahl's figure 1), although the area cannot be clearly demarcated in Musgrave's (as well as in Lindahl's) partial equilibrium diagram.

Musgrave also thought that the solutions corresponding to the concept of the contract curve or "area," that is, "a large number of points movement between which involves a gain to either A or B and a loss to the other" (p. 79), lay on or to the left of \(UDY\) in his figure 4-2. This is incorrect. In his diagram, the contract curve lies to the right of \(UDY\) and to the left of \(b_1DA_1\) and cuts two demand curves at point \(D\), as can be inferred easily from our figure 5.

Having come to this stage, it is not difficult to see that a variety of other behavioral assumptions can be introduced into the analysis. One such assumption, which has attained some popularity of late, is that negotiation on the distribution of costs of public expenditure is made by offers and counteroffers in terms of absolute contributions that each party makes (and not in terms of cost-share ratios). This assumption

\[21\] Johansen (1963) was the first person ever to specify diagrammatically a part (but not the whole, although he explicitly suggested the possibility of extending his diagram to include the whole) of the contract curve, and his success was followed by Campa (1967). But both failed to uncover and delimit the trading area; the tasks apparently have been left to the present paper. Dolbear's contribution (1967), although his was in the context of external diseconomy, should be noted, however.
was employed recently by Williams (1966), Olson and Zeckhauser (1966), and Campa (1967). The assumption relies on the fact that, for any given contribution offer of one party, there is an optimum amount of public expenditure to which the other party agrees. In figure 6, reproduced from figure 4, a given amount of contribution toward the costs of the public good \((X_a)\) that party \(A\) offers is indicated by the distance from \(D\) to the origin of party \(A\) on the \(AB\) axis, say \(D_1D\). In other words, party \(A\)'s offer amounts to an increase in party \(B\)'s pretax income by that amount. The optimum size of public expenditure for party \(B\) when party \(A\)'s contribution is \(D_1D\) is indicated, of course, by a point where one of party \(B\)'s adjusted indifference curves is tangent to the new "budget" line of party \(B\) or the horizontal line containing point \(D_1\). Gradually varying the absolute amount of tax that party \(A\) offers and locating a series of new tangency points, we can generate party \(B\)'s reaction curve, \(SS\), which is analogous to the line commonly called the income-consumption curve, or Engel curve. Similarly, we can obtain party \(A\)'s reaction curve \(RR\) by shifting party \(A\)'s "budget" line, a line parallel to \(A\)'s production-possibility curve \(DE\), and tracing the locus of tangency points between party \(A\)'s indifference curves and these various budget lines. The intersection of these two reaction curves necessarily determines an equilibrium solution, which is stable unless the public good is an inferior good and as long as the two parties behave as assumed.

But at this intersection point, the slopes of the indifference curves of the two parties differ: The slope of party \(A\)'s indifference curve is the same as the slope of \(BC\), and that of the adjusted indifference curve of party \(B\) is zero. Therefore, this solution is not on the contract curve, as is clearly shown in figure 6. Since there must be considerable room left for the two to improve their welfare simultaneously without hurting either, it is doubtful whether both parties will remain satisfied at this solution forever. It is more likely that they will change their expectations as to the behavior of the other party; and finding a way to accommodate their mutual interest, they will move toward one of the

22 Campa's statement "for any given \(B\)'s offer, there is a maximum amount of \(G\) to which \(A\) agrees" (1967, p. 404) must be an error. I assume that he meant "an optimum amount," instead.

23 This conclusion is the same as those reached by Williams (1966), Olson and Zeckhauser (1966), and Campa (1967), but their analytical techniques are different from ours. In their approaches, the costs of public expenditure are expressed in terms of money value, as was the case of their apparent analytical predecessor, Johansen (1963). Therefore, their diagrammatic analyses, including Johansen's, must assume constancy of opportunity cost and are not readily amenable to the analyses of increasing (or variable for that matter) marginal costs cases. Our analysis measures the costs of public expenditure in terms of private goods forgone and, consequently, has the advantage of being easily extendible to the analyses of variable marginal opportunity cost cases, as will be shown in Section III.
Pareto-preferred solutions of that temporary equilibrium solution (the shaded area of fig. 6). But to ascertain the location of the final solution, the information at our disposal is again not sufficient.

III. The Case of Increasing Marginal Costs

Our analysis has proceeded so far on the assumption that factor intensities in production of X₁'s and Xₙ are equal. The assumption is now dropped in order to analyze income-redistribution effects consequent upon the production of public goods.

When factor intensities differ between X₁'s and Xₙ, the production-possibility curve exhibits concavity from below to reflect the usual assumption of increasing relative marginal costs as shown in figure 7. Using the same technique explained in Section I, we can generate party B's map of adjusted indifference curves from figure 2 for this case in which the rotated axis has the shape of the inverted production-possibility curve, and when it is combined with figure 1, they produce figure 7. Convexity of the adjusted indifference curves of B is more pronounced in this case than in the constant-cost case although, nevertheless, the general nature of the box diagram–cum–transformation curve diagram remains the same. The contract curve is represented as before
by the locus of tangency points between A's and B's (adjusted) indifference curves, VW. The trading area can generally be delimited as before as the area enclosed by the boundaries consisting of two indifference curves, each of which is the highest of the respective party's own indifference curves that each party can attain independently, which is usually the one tangent to that party's "independently attainable consumption-possibility frontier."

We must now, however, identify each party's independently attainable consumption-possibility frontier. If the members of party A and party B draw their respective incomes in equal proportions from various factors, changes in relative factor earnings resulting from changes in the product mix will not alter the income share between the parties. Therefore, in figure 7, if the initial income distribution between the parties is given as indicated by AD/DB, the independently attainable consumption frontiers of both parties originate at point D and that of party B is a horizontal line, DF, and that of party A is obtained as a curve, DE, produced by shifting the production-possibility curve, BC, vertically downward until point B coincides with point D. In the case of figure 7, since both parties' indifference curves tangential to their respective independently attainable consumption-possibility frontiers

![Fig. 7](image-url)
indicate higher levels of welfare than those representing their initial positions, the trading area is the area demarcated by the former indifference curves, namely, $RY'$ and $SZ'$. In this case, the trading area is necessarily a proper subset of the Pareto-preferred solutions of the initial position $(D)$, and the conclusions of the previous analysis apply equally well to this case.

If, however, our rather artificial assumption on the distribution of ownership of factors of production is dropped, we must allow for changes in the distribution of income accompanied by changes in relative factor earnings as product mix changes. The party whose members own relatively more of the factor used intensively in producing the public good will increase its income as a result of an increase in production of the public good, and the party whose members own relatively less of that factor will suffer a loss of income. The net change in a party’s welfare resulting from a change in the output of the public good is, therefore, divided into two main parts: the one which is due to a change in the income distribution, which may be called the income-redistribution effect, and the other, due to change in the consumption opportunity of the public goods, which may be called the exchange-opportunity effect, which, in turn, is as usually divided into the income effect and the substitution effect. In this case, the contract curve will remain unchanged, but the determination of the trading area is a little more complicated because independently attainable consumption frontiers will shift as the product mix changes.

Let us assume that the members of party $A$ as a whole relatively less of the factors used intensively in producing the public good and the members of party $B$ as a whole relatively more of such factors; and, as a result, as the output of the public good ($X_n$) expands, the share of party $A$ in the economy’s total income declines. This is shown in figure 8 by the downward movement of the point indicating income-distribution ratio, or point $D$. Thus, $AD_1/AB$, $AD_2/AB$, $AD_3/AB$, . . . , represent party $A$’s share of the total income when the output of $X_n$ is $AG_1$, $AG_2$, $AG_3$, . . . , respectively. Therefore, when the output of $X_n$ is $AG_1$, and if party $A$ has to bear the entire tax bill, $N_1B$, it can consume $AG_1$ of $X_n$ and $G_1L_1$ ($=AD_1 - N_1B = AD_1 - L_1K_1$) of $X_j$. Similarly, when the public-good output is increased to $AG_2$, $AG_3$, $AG_4$, . . . , and party $A$ is assumed to remain the sole bearer of the entire tax bill, its consumption opportunity is represented by points $L_2$, $L_3$, $L_4$, . . . , respectively. The locus of these points, $DE$, is, of course, party $A$’s independently attainable consumption-possibility frontier. On the same diagram, point $K_1$ represents the consumption opportunity of party $B$ when it bears the entire cost of $AG_1$ quantity of the public good. It consists of $AG_1$ of $X_n$ and $K_1O_1$ ($=D_1B - N_1B$) of $X_j$. Similarly, party $B$’s consumption
opportunities attainable independently when $X_a$ outputs are $AG_2$, $AG_3$, $AG_4$, ..., are represented by $K_2$, $K_3$, $K_4$, ..., respectively, and the locus of these points, $DF$, is party $B$'s independently attainable consumption-possibility frontier. If factor-intensity reversals in production take place, these frontiers will "fluctuate." Nevertheless, for simplicity, in the following we assume throughout that the two frontiers are monotonically changing, negatively sloped, and concave from below. By finding party A's (B's) indifference curve tangential to its independently attainable consumption-possibility frontier (permitting a corner solution), we name it $RY''$ ($SZ' $), which is obviously the highest possible welfare level that party $A$ ($B$) can attain independently from the whole possible range of output mixes. Clearly, in the situation as described in figure 8, to reach an indifference curve higher than $RY''$ or $SZ'$, each party must share the tax bill with the other; thus the area demarcated by these two indifference curves is the trading area.

24 The locus of $K_1$, $K_3$, ..., is also a diagrammatic representation of the proportion of party $A$'s pretax income in the group's total income as a function of the quantity of the public good produced.
In the variable marginal cost case, however, under some circumstances the area thus delimited may be a null set. In other words, there may be no trading area which is a proper subset of the area of Pareto-preferred solutions of the initial position, represented by point D. Namely, depending upon the given state of distribution of ownership of factors of production and the given technological condition, the income-redistribution effect may become so unfavorable to one party that, even if the other party bears the entire tax bill, production of a public good will place the unfavorably affected party on an indifference curve lower than its original one. This case is illustrated in figure 9 where party B's independently attainable consumption-possibility frontier $DF$ is below party A's indifference curve representing its initial welfare position, $DY$, for the entire possible range of $X_n$ output. Under such a circumstance, the unfavored party (party A) could be placed on lower and lower indifference curves as the public good output expands, depending on the relative positions and the shapes of the unfavored party's indifference curves and the favored party's independently attainable consumption-possibility frontier ($DF$), unless the unfavored is given a sufficient amount of compensation in terms of the private good from

![Figure 9](image-url)
the favored. However, there is nothing in our model to stop a utility-maximizing party (party B) from expanding output of a public good by its own finance up to its saturation point, where normally one of its indifference curves (SZ’) is tangent to its own independently attainable consumption-possibility frontier (DF). Such a point is denoted by S in figure 9.25 Nor is there a mechanism that enables the unfavored party A to extract “compensatory payments” from the favored. It is only when the favored reaches its own saturation point, S, that it begins to prepare itself to enter “negotiation” with the unfavored. Therefore, the trading area is the area demarcated by the two parties’ indifference curves containing point S. Thus, in this case, the trading area and the Pareto-preferred solutions of the initial position are mutually exclusive sets.26

This last case clearly reveals one of the basic difficulties of the so-called voluntary-exchange theory of public finance. The key notion behind the theory is obviously an analogy with a pure-exchange market situation in which, if one enters into agreement freely with the other about the exchange of particular goods and services, this must imply an improvement of the welfare of both as compared with their initial positions: Otherwise they would not enter into that agreement. A corollary of this is that not entering into a particular exchange may not improve one’s position but at least it will not worsen it. Similarly, it might be thought that if the institutional framework of negotiation is maintained and the provision of public goods is left to voluntary action, it will guarantee that no one will be hurt and someone may gain. This argument would be correct if the bargaining power worked in both positive and negative directions in negotiating individuals’ share of contributions toward the public expenditure. However, the bargaining power based on an ability to withhold contribution toward the public funds becomes powerless when it must be used to extract compensatory payments from (or to contribute negatively to) the public funds. Our analysis above demonstrates clearly that, where provision of public expenditure entails a considerable change in factor earnings, not contributing to the public funds at all (and enjoying of course free public services) is not adequate

25 The S could be a corner solution, however.

26 The point S indicates a distinct possibility that a public good could become too large (that is, it is expanded beyond its Pareto-optimum level) when the income-redistribution effect occasioned by the production of that public good is large and the possibility of negotiation between the parties affected by the provision of that good is severely limited. Indeed, the force of the income-redistribution effect accompanying the provision of public goods is a rather familiar ingredient of the politics of public works. Nevertheless, we cannot generalize this possibility to a case where bargaining between the parties is fully possible, for, as discussed in Section I, bargaining could end with a solution located either to the left or the right side of (or right on) the contract curve. The inference above, therefore, should apply more appropriately to the case of a good imposing an external (economy) effect where no bargaining possibility exists.
for the unfavorably affected party to prevent its own initial welfare position from worsening although it is the policy that maximizes its welfare position.

The asymmetry of bargaining power implies that, even if the initial income distribution is just, the voluntary-contribution approach not only changes unevenly the welfare positions of the parties concerned but also may absolutely worsen one party’s position as compared with its initial position. Thus, in general, even if the negotiation is carried out up to “saturation”—in Lindahl’s sense—and an efficient solution is obtained, the initially just distribution of income does not guarantee that the solution is even a Pareto-preferred solution of the initial position—not to speak of the social welfare maximum. By itself, the voluntariness of the system is not a sufficient condition to assure all the parties concerned against possible welfare losses. To make the voluntary-contribution system work as a mechanism to achieve always a Pareto-preferred solution, there is a need to redress the balance of bargaining powers so that the party unfavorably affected can bargain for negative contributions or for subsidies.

This defect of the voluntary-contribution approach does not constitute a factor which favors the Wicksellian unanimous-voting approach, however. As quoted earlier, Wicksell proposes to judge worthiness of one public expenditure, a new activity or an extension of the old, by the possibility of distributing its “direct costs” among voters in such a way that everyone gains from the introduction of that expenditure. Evidently, he paid attention only on the task of distributing the direct costs of public services and failed to recognize possible voters’ reactions to the income-redistribution effects which almost any sizable public expenditure must necessarily entail. It is clear from our preceding analysis that, under his scheme, even in the absence of the game-theoretic strategies, many new expenditures (or extensions of the existing programs) which accompany relatively strong income-redistribution effects stand little chance of being unanimously approved, even though they are perfectly worthy expenditures in the sense that they are capable of making everyone better off if the distribution of negative tax (or compensatory payments) is also effected. As already shown, the assumption of an initially optimum (or just) distribution of income does not solve this problem. This leads to a conclusion that, under the present institutional setting, to reach even a simple Pareto-preferred solution, not to speak of the grand social optimum solution, a voluntary provision of a public good often must simultaneously be accompanied by an involuntary redistribution of income from the apparent cost bearers to the apparent “free riders.”

27 This point was stressed by Myrdal (1930) and Samuelson (1955).
References


