Majority voting with single-crossing preferences

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Abstract

We clarify and extend a number of sufficient conditions for the existence of a majority voting equilibrium on one-dimensional choice domains. These conditions, variously stated in the previous literature, all impose order restrictions on voter preferences (for instance, monotone marginal rates of substitution) which we show to imply or be equivalent to a general, ordinal version of the single-crossing condition. This simple property is economically intuitive and easily checked in applications. This ease of application is demonstrated through an examination of voting models of redistributive income taxation and trade union bargaining behaviour.

Keywords: Majority voting; Single-crossing condition; Quasi-transitivity; Income tax progression; Union objectives

JEL classification: D71; H20; J51

1. Introduction

A number of recent papers have provided conditions that are sufficient to ensure the existence of majority voting equilibrium, but which are, nonetheless, distinct alternatives to single-peakedness. These conditions, while variously stated, are essentially of two types. The first type reduces to a single-crossing requirement on the indifference curves of voters.\textsuperscript{1} That is,

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\textsuperscript{1} See, for example, Eppie and Romer (1991), Westhoff (1977), and others.
marginal rates of substitution are restricted to be monotone in some order of the voters at all points in the choice domain. The second type consists of more abstract conditions requiring a global ordering of some representation of voters' preferences. In this paper we unify these results by showing their relationship to the general, ordinal notion of single-crossing proposed by Milgrom and Shannon (1994), which is shown to be sufficient for a majority voting equilibrium to exist. That condition is itself related to the more familiar Spence–Mirrlees single-crossing, or sorting, condition that is invoked frequently in several fields related to voting theory, including mechanism design, principal–agent theory and information economics.

In particular, in the main result of the paper we show that if the set of alternatives is one-dimensional and if preferences satisfy the single-crossing condition as defined by Milgrom and Shannon, then majority voting induces a quasi-transitive social preference over the set of alternatives. Indeed, a stronger version of the Milgrom–Shannon condition is sufficient for transitivity of majority preferences, under some additional restrictions.

As well as highlighting the potential importance of single-crossing preferences in social choice theory, our method of proof is considerably simpler and more intuitive than related results. The primary virtue of stating a very intuitive sufficiency condition lies in its ease of application to voting problems frequently studied in economics. Indeed, we provide a number of straightforward 'tools' for testing whether single-crossing holds in economic applications of interest. Some prominent examples of these are explored in the subsequent sections of the paper.

In Section 3 we examine a model of a society voting over distortionary redistributive tax schedules. We show that, in the case of two-part linear tax schedules, single-crossing is equivalent to the Hierarchical Adherence condition of Roberts (1977) and provide an alternative proof of Robert's results. We then extend the result to any family of non-linear tax schedules of increasing progressivity.

Another common application of voting models is to the objectives of democratic labour unions. In Section 4 we consider the model of a 'monopoly' (wage-setting) union constructed by Blair and Crawford (1984), in which union members have heterogeneous preferences over wages and employment levels, due to their differing levels of seniority in hiring decisions. We show that union objectives are well defined under conditions different and perhaps more reasonable than those postulated by Blair and Crawford. Moreover, under single-crossing the union's preferences can be represented by a cardinal (von Neumann–Morgenstern) utility function as long as the median voter's preferences can be. In contrast, under

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2 See, for example, Grandmont (1978), Roberts (1977), and Rothstein (1990, 1991). Our work is closest to that of Rothstein, and some of our results have parallels in his work.
single-peakedness cardinality cannot be shown to be inherited by the union's objective function.

2. The theorems

We consider a set of social alternatives $X$ and a set of voters $\Theta$. Unless otherwise indicated, we require that both sets be chains; that is, for each set there is a weak order $\succeq$ such that for all $x, y \in X$, either $x \succeq y$ or $y \succeq x$. Similarly, for all $\theta, \theta' \in \Theta$, either $\theta \succeq \theta'$ or $\theta' \succeq \theta$. (Although the orders on the two sets need not be related, we will use the same symbol for both without much danger of confusion). The asymmetric factor of $\succeq$ is as usual denoted $\succ$. Often, in applications, $X$ will be a subset of the real line and $\succeq$ will be the usual order on $R$.

In addition, we will assume that $\Theta$ is a finite index set of natural numbers, with cardinality $N$; thus $\Theta = \{1, \ldots, N\}$. Extending the results to infinite but measurable populations, which is technically convenient in a number of applications, is a straightforward task.$^3$

The preferences of each individual $\theta$ are given by some complete, reflexive binary relation $R_{\theta}$ on $X$. The corresponding strict preference relation, the asymmetric factor of $R_{\theta}$, is denoted $P_{\theta}$ (Note that transitivity is not required at this point.) A preference profile is denoted by $R_{\Theta}$.

Majority rule is defined as follows. Let $\mu(x, y)$ be the fraction of voters $\theta$ in $\Theta$ for whom $x P_{\theta} y$. Then $x$ is preferred to $y$ under majority rule (denoted $x R^M y$) if and only if $\mu(x, y) > \mu(y, x)$. The relation $R^M$ will be termed the majority preference relation.

2.1. Single-crossing and majority preferences

A preference profile $R_{\Theta}$ is said to satisfy the single-crossing condition (SC) if

$$\forall x' > x, \forall \theta' > \theta : x' R_{\theta} x \Rightarrow x' R_{\theta} x \text{ and } x' P_{\theta} x \Rightarrow x' P_{\theta} x.$$ 

In some cases, a stronger condition – the strict single-crossing condition (SSC) – will be required:

$$\forall x' > x, \forall \theta' > \theta : x' R_{\theta} x \Rightarrow x' P_{\theta} x.$$ 

When $N$ is even, we define the median voters by $\theta_1 = N/2$ and $\theta_2 = N/2 + 1$, and denote their corresponding preferences by $R_1$ and $R_2$. When $N$ is odd, $\theta_1 = \theta_2 = (N + 1)/2$. This approach is somewhat unusual but allows us

$^3$ This extension is contained in an older version of this paper, Gans and Smart (1994).
to simplify the ensuing discussion without restricting attention unnecessarily to the case of an odd number of voters.

With this terminology in place, it can be shown that when preference profiles are single-crossing the median voters in the order on \( \Theta \) are decisive in all majority elections between pairs of alternatives in \( X \).

**Theorem 1.** Suppose that \( X \) and \( \Theta \) are chains and preferences satisfy SC. Then for all \( x, y \in X \):

(i) \( xR^M y \) if \( xP_1 y \) or \( xP_2 y \);

(ii) \( xP^M y \) if \( xR_1 y \) and \( xR_2 y \) with at least one strict preference.

All results are proved in the appendix. Since the median voters are decisive in all cases in which they have a common strict preference over a pair of alternatives, it is possible to exploit the transitivity of individual preference relations to prove that the majority preference relation is quasi-transitive.

**Corollary 1.** Suppose that the conditions of Theorem 1 hold and that the preference profile is transitive. Then \( R^M \) is quasi-transitive.

Complications arise when the median voters are indifferent between a pair of alternatives, since they need not be decisive under SC in these cases. When SSC obtains, however, the median voters are decisive over all pairs, and a complete characterization of the majority preference ordering is available.\(^4\)

**Theorem 2.** Suppose that \( X \) and \( \Theta \) are chains and preferences satisfy SSC. If there are two median voters with distinct preferences, then \( xR^M y \) if and only if \( xP_1 y \) or \( xP_2 y \). If there is only one distinct median voter \( \theta_1 \), then \( xR^M y \) if and only if \( xR_1 y \).

Since in this last case the majority preference relation and the preferences of the unique median voter coincide, they have all the same properties. This fact is convenient in a number of applications.

**Corollary 2.** Suppose that the conditions of Theorem 2 hold and there is a unique distinct median voter. Then \( R^M \) inherits all the properties of \( R_1 \). In particular, \( R^M \) is transitive if and only if \( R_1 \) is.

In the course of our research, it was discovered that a similar set of results was derived by Rothstein (1990, 1991) under a sufficient condition which he

\(^4\) We are indebted to Paul Milgrom for this observation.
called order restriction. This restriction is essentially equivalent to our ordinal single-crossing condition defined above. This equivalence will be demonstrated here for the sake of completeness.

The following definition is equivalent to Rothstein’s condition, but more consistent with our notation. The indifference relation associated with \( R_o \) is denoted \( I_o \). We say a preference profile \( R_o \) satisfies order restriction (OR) if for all \( x, y \in X \) either

\[
\{ \theta : xP_\theta y \} > \{ \theta : xI_\theta y \} > \{ \theta : yP_\theta x \} \tag{OR-1}
\]

or

\[
\{ \theta : yP_\theta x \} > \{ \theta : yI_\theta x \} > \{ \theta : xP_\theta y \}, \tag{OR-2}
\]

where \( > \) is the strict set order defined by \( S > S' \) if for all \( x \in S \) and \( y \in S', x > y \).

**Theorem 3.** The preference profile \( R_o \) satisfies OR if and only if it satisfies SC.

An immediate corollary of the theorem is that the results of Rothstein, obtained under order restriction, also hold under the single-crossing condition. In particular, Rothstein (1990) investigates the relationship between the domain restrictions of Pattaniak and Sen (1969) and OR (and hence SC as well). With his results, it follows that if the preference profile satisfies SC on a given triple of alternatives \( \{x, y, z\} \subset X \), then preferences satisfy value restriction on that triple. The converse, however, need not hold. Since value restriction is a sufficient but not necessary condition for quasi-transitivity of the majority preference in the work of Pattanaik and Sen, it follows immediately that SC is sufficient but not necessary as well. Indeed, SC should be regarded as a fairly strong sufficient condition that rules out ‘many’ profiles that do yield quasi-transitivity.

A version of Theorem 1 is established in Rothstein (1991) for the order restriction condition. He also shows that OR implies the acyclicity of majority strict preferences, which is a weaker version of our Corollary 1, which proves quasi-transitivity. Our transitivity and other inheritance results, established in Corollary 2, are paralleled in Rothstein’s Theorem 3, which assumes that all voters’ preference orders are strict. The assumption of strict preferences, which is not useful or interesting in economic environments, is replaced in our analysis by the strict single-crossing condition.

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\(^5\) A closed preference order on a real-valued domain is strict only if it is strictly monotone. In this case, the majority preference is in fact a consensus preference.
2.2. Characterizations of single-crossing

Previous research has provided a number of sufficient conditions for the existence of majority voting equilibria that have had little intuitive economic content and which have, as such, been difficult to verify in economic applications. Given our fairly abstract statement of the single-crossing condition above, it might seem that our results suffer from the same problem. Several recent papers have, however, yielded other characterizations of single-crossing that are more familiar and hence more amenable to analysis.6

The most salient of these results is the close connection between single-crossing and monotone comparative statics in optimization problems, which is proven in Milgrom and Shannon (1994). In order to analyze optimal choices, we must additionally require that preferences \( R_\theta \) be closed and transitive. With these assumptions, there exists a numerical representation \( u(\cdot, \theta) \) of \( R_\theta \). We then say the function \( u \) satisfies SC in the pair of variables \((x, \theta)\) if the preference profile it represents satisfies SC on \( X \), as defined in the previous section.8

Some additional notation is required. For any set \( X \) and \( x, x' \in X \), \( x \land x' \) denotes \( \inf \{x, x'\} \) and \( x \lor x' \) denotes \( \sup \{x, x'\} \). A set \( X \) is a lattice if \( x, x' \in X \Rightarrow x \lor x', x \land x' \in X \). A function \( f: X \rightarrow \mathbb{R} \) is quasisupermodular if for all \( x, x' \in X \), \( f(x) \geq f(x \land x') \Rightarrow f(x \lor x') \geq f(x') \).8 For any two sets \( S, S' \subset X \), the strong set order \( \geq_s \) is defined by: \( S' \geq_s S \) if for all \( x' \in S' \) and \( x \in S \), \( x \lor x' \in S' \) and \( x \land x' \in S \).

Lemma 1 (Milgrom and Shannon, 1994, Theorem 4). Let \( u: X \times \Theta \rightarrow \mathbb{R} \), where \( X \) is a lattice and \( S \subset X \). Then

\[
\arg \max_{x \in S} u(x, \theta)
\]

is weakly increasing in \((\theta, S)\) if and only if \( u \) is quasisupermodular in \( x \) and satisfies SC in \((x, \theta)\).

6 See, inter alia, Shannon (1992), Milgrom and Shannon (1994), Milgrom (1994), Milgrom and Roberts (1994) and Athey (1994). These works contain a number of other characterization results not addressed in this paper, which may be useful in voting models. Here we focus on the essential tools for establishing single-crossing in voting models.

7 The original definition of single-crossing in Milgrom and Shannon (1994) assumed the existence of a utility function rather than a binary preference. With this additional assumption, our definition and theirs are equivalent.

8 Since in our applications \( X \) will be a chain, the operations are simply the maximum and minimum of the two points, respectively, and any subset of \( X \) is a lattice. If \( X \) is a chain, moreover, then any function on \( X \) is quasisupermodular. Finally, a function is quasisupermodular in \((x, y)\) if it satisfies the single-crossing property in both \((x, y)\) and \((y, x)\). The more general approach is used in the lemmata below, however.
In the context of majority voting when $X$ is a chain, SC admits a consistent order of voters’ most-preferred alternatives from any subset of $X$. In contrast, while single-peakedness is preserved on any subset of $X$, the order of voters’ most-preferred alternatives may differ on different subsets. This is the essential difference between the two conditions.

Ordinal single-crossing is, of course, closely related to the more familiar Spence–Mirrlees condition that agents’ marginal rates of substitution be ordered. In some cases, the two conditions are equivalent. This equivalence holds when underlying preferences are defined over a two-dimensional real choice variable, but—given budget or production constraints on feasible choices, for example—attention can be restricted to a one-dimensional manifold in $R^2$.

Let $X \subseteq R^2$ and let preferences be represented by a continuously differentiable utility function $u: X \times \Theta \rightarrow R$ with $u_{\theta}(x_1, x_2, \theta) > 0$ for all $(x, \theta)$ (subscripts of functions denote partial derivatives). Suppose additionally that the levels sets of $u$ are curves in $X$. We say $u$ satisfies the (strict) Spence–Mirrlees condition on $S \subseteq R$ if for all $x_1 \in \text{int} S$ and for all $x_2 \in R$, the marginal rate of substitution $\sigma(x, \theta) = u_1(x, \theta)/u_2(x, \theta)$ is (strictly) increasing in $\theta$. With this definition, we have the following result, which is due to Milgrom (1994).

**Lemma 2** (Milgrom, 1994, Theorem 1). Let $X \subseteq R^2$ and $u_{\theta} > 0$. The function $u(x_1, f(x_1), \theta)$ is (strictly) single-crossing in $(x_1, \theta)$, $x_1 \in S$, for all functions $f$ if and only if $u$ satisfies the (strict) Spence–Mirrlees condition on $S$.

Thus, in many applications of the theory it will be sufficient to check that marginal rates of substitution are monotone in voters’ types to ensure that the conditions of the theorem obtain. In such cases, SC can be reduced to intuitive conditions on the economic trade-offs facing voters.

In some cases of interest, voters’ preferences are defined over a subset of variables that are private goods, chosen in a decentralized manner by each voter independently, and a subset that are public goods, chosen by majority election. In such cases our sufficient condition will require that voters’ preferences be single-crossing, after optimizing over private choices given the public decisions taken under majority rule.\(^9\)

Consider an environment with preferences $u: X \times Y \times \Theta \rightarrow R$, where $X$ and $Y$ are chains representing feasible choices of public and private goods, respectively.\(^10\) Voters choose $y \in Y$ in a decentralized fashion given a social

\(^9\) This is the ‘aggregation principle’ of Milgrom and Shannon (1994).

\(^10\) The result below can be extended to the case of many private goods $y$, in the manner of Lemma 1.
choice \( x \). According to Theorem 1, a sufficient condition for the majority preference relation to be quasi-transitive is that the envelope function

\[
v(x, \theta) = \max_{y \in Y} u(x, y, \theta)
\]

be single-crossing in \((x, \theta)\). A number of different conditions on the primitive preference \( u \) are sufficient for \( v \) to be single-crossing in \((x, \theta)\), each of which may be useful in particular applications. Two such alternative sufficient conditions are presented in the following lemma.

**Lemma 3.** Let \( X \) and \( Y \) be chains. Then \( v \) is single-crossing in \((x, \theta)\) if either of the following conditions holds.

(a) \( u \) is single-crossing in each of the pairs of variables \((x, \theta)\) given \( y \) and \((y, \theta)\) given \( x \), and \( u \) is quasisupermodular in \((x, y)\) given \( \theta \).

(b) There exist functions \( U \) and \( h \) such that

\[
u(x, y, \theta) = U(h(x, y), y, \theta),
\]

where \( U(z, y, \theta) \) is twice differentiable in \((z, y)\) and strictly increasing in \( z \), \( U_y/U_z \) is increasing in \( \theta \), and \( h \) is single-crossing in \((x, y)\).

The intuition for the results is straightforward. To establish that \( v \) is single-crossing we require that increasing \( \theta \) alone strengthens the preference for each of \( x \) and \( y \), and that increasing \( y \) strengthens the preference for \( x \) in turn. Each part of the lemma provides an alternative sufficient condition for ensuring that the 'feedback' effect of the change in preference for \( x \) does not undermine the stronger preference for \( y \). In part (a), quasisupermodularity is imposed, so that \( x \) and \( y \) are directly complementary. In part (b), \( x \) is required to be weakly separable from \( \theta \), so that feedback effects are limited. (Often, in applications, the aggregator function \( h \) has an interpretation as a budget constraint, so that the result is essentially a multivariate version of Lemma 2.)

### 3. Application I: Voting over tax schedules

A classical application of the majority voting model is to the problem of a society of heterogeneous voters choosing a schedule of redistributive tax and transfer rates. Previous research has examined in particular redistributive income taxation and its impact on labour supply decisions, or on location decisions in models of local public finance. Because, in both cases, aggregate responses to the tax policy are of critical importance, voters' preferences over tax schedules may not be single-peaked, raising the possibility that the majority preference relation be intransitive and that voting equilibria fail to
exist. Moreover, violations of single-peakedness can occur even when partial-equilibrium individual preferences – taking the behaviour of other voters as given – are single-peaked and well-behaved in every respect.

While this observation may seem contradictory, the problem can be readily discerned in an example. Imagine a linear income tax schedule that is levied by government to finance a guaranteed income policy or refundable tax credit. Beginning from any status quo policy, a voter with low market income may prefer to raise tax rates, taking labour supply decisions of other earners as given, since this increases the lump-sum transfer that is feasible under the government’s budget constraint. When aggregate labour supply is fixed, therefore, it follows that each voter has a single most-preferred tax rate, determined by the voter’s ranking in the pre-tax distribution of income, and that preferences over all feasible tax rates are quasi-concave. But if the increase in the marginal tax rate induces high-income earners to work less then tax revenue and lump-sum transfers may decline, so that low-income earners locally prefer lower tax rates. Unless aggregate income tax revenue is a concave function of tax rates (and there is no compelling reason to believe it should be), indirect preferences over tax rates need not be single-peaked. Consequently, the existence of a quasitransitive majority preference relation cannot be guaranteed.

Roberts (1977) examined the problem just described and showed that if individuals’ gross labour incomes can be ranked independently of the tax policy adopted – a condition he called Hierarchical Adherence – then a majority voting equilibrium exists. Roberts proceeded by demonstrating that the Hierarchical Adherence condition implied that voters’ preferences satisfied value restriction as defined by Pattanaik and Sen (1969). Using our techniques, however, it can be shown far more easily that Roberts’s condition implied single-crossing.

Individuals are endowed with labour which they supply to the market to finance purchases of a single consumption good. Underlying preferences are represented by a utility function $u(c, y, \theta)$, where $c$ and $y$ are the individual’s consumption and labour supply measured in units of pre-tax earnings, respectively, and $\theta$ is an index representing the individual’s ability. The utility function is taken to be continuously differentiable with $u_c > 0$.

Tax schedules consist of a lump-sum transfer $G$ to all taxfilers and a constant marginal tax rate $t$ on labour income. An individual’s consumption is therefore

$$c = G + (1 - t)y.$$  \hspace{1cm} (1)

Individual preferences over tax policies are

$$v(G, t, \theta) = \max_{y \in Y} u(G + (1 - t)y, y, \theta),$$  \hspace{1cm} (2)
where $Y$ is any compact set of positive reals [which may depend on $(G, t, \theta)$]. We restrict attention to self-financing policies, so that the feasible per capita transfer is given by some function $G(t)$.

It follows from Lemma 2 above that voters' preferences over feasible tax schedules, $v(G(t), t, \theta)$, are single-crossing in $(t, \theta)$ if $v(G, t, \theta)$ satisfies the Spence–Mirrlees condition; namely if voters' marginal rates of substitution between $G$ and $t$ are globally increasing in $\theta$. By the envelope theorem,

$$ \frac{\partial u(G + (1-t)y^*, y^*, \theta)}{\partial G} = -\frac{\partial u(G, t, \theta)}{\partial G} = y^*(G, t, \theta), $$

where the derivatives are evaluated taking $y^*$ fixed at its optimal level. Hence the Spence–Mirrlees condition holds if and only if $y^*(G, t, \theta)$ is increasing in $\theta$, which is precisely Roberts's Hierarchical Adherence condition.

Proposition 1. Suppose that majority rule is used by society to choose a two-part linear income tax schedule and that labour-supply behaviour is such that there exists a functional relationship between marginal tax rates and gross tax revenue. Then a sufficient condition for quasitransitivity of the majority preference relation is that optimal labour supply be increasing in some invariant order of the voters for all possible tax schedules.

The result required no condition on the aggregate response function $G$, other than it simply be a function.\(^{11}\) This fact, which is one of the most fundamental notions in the monotonicity analysis of Milgrom and Shannon (1994), allows the theorem to be applied in many cases in which other approaches are not useful.\(^{12}\)

Given the ordinal nature of our sufficient condition, the result can be generalized considerably to apply to elections over any family of non-linear tax schedules that can be ordered by a notion of progressivity. Thus Roberts' existence result did not depend upon the linearity of the tax schedules considered, but rather on the fact that the constant marginal tax rate indexed the progressivity of the tax policy in a simple way.\(^{13}\)

\(^{11}\) Restricting $G$ to be a function is an exceptionally mild condition, which holds if, for example, consumption is normal for all workers.

\(^{12}\) For example, strong convexity conditions were required in Westhoff's (1977) analysis of redistribution and mobility in local public finance. Epple and Romer (1991) used a condition essentially equivalent to SSC to show that the convexity assumptions were unnecessary in such contexts. We thank an anonymous referee for pointing this out.

\(^{13}\) Our results also generalize the work of Cukierman and Meltzer (1991), who consider linear-quadratic tax schedules. Restricted classes of non-linear tax schedules are also considered by Berliant and Gouveia (1994), where different techniques of proof are employed and broader questions of existence are addressed.
For a particular tax policy \( \tau \), let \( n(y, \tau) \) represent the net (after-tax) income of any individual with gross income \( y \). Suppose that \( n(y, \tau) \) is continuous with \( y \). We wish to restrict attention to sets of tax schedules completely ordered by the following definition of progressivity: a schedule \( \tau' \) is said to be more progressive than a schedule \( \tau \) if the after-tax distribution of income induced by \( \tau' \) Lorenz dominates that induced by \( \tau \) for all pre-tax distributions of income such that the two schedules raise the same aggregate revenue. It was established by Hemming and Keen (1983) that this notion of progressivity is equivalent to the requirement that net income \( n \) be single-crossing in \( (\tau, y) \). In this case, single-crossing means that for any two tax schedules in the family, if an individual of gross income \( y \) receives higher net income under the less progressive of the two schedules, then all individuals with gross income \( y' > y \) will also have higher net income under the less progressive schedule. With this condition, it can be said unambiguously that tax progressivity is decreasing in the parameter \( \tau \).

We additionally impose the Hierarchical Adherence condition that \( u(n(y, \tau), y, \theta) \) be single-crossing in \( (y, \theta) \), so that the individuals can be ordered by their gross incomes independently of the tax policy adopted. To see the role of this assumption, consider the simplest case in which individuals have identical preferences over consumption-leisure bundles, but individuals of higher ability \( \theta \) earn higher wages. Then Hierarchical Adherence is satisfied as long as consumption is normal in utility. In this context, the assumption is merely the usual sufficient condition for the incentive compatibility of the tax schedule.

Preferences over tax schedules are

\[
v(\tau, \theta) = \max_{y \in Y} u(n(y, \tau), y, \theta).
\]  

(1)

Note that \( n \) is single-crossing in \( (\tau, y) \) by the progressivity condition and single-crossing in \( (y, \theta) \) by Hierarchical Adherence. Thus the conditions of Lemma 3(b) obtain, and it follows that individuals' preferences over tax schedules are single-crossing, which yields the following result.

**Proposition 2.** Suppose voting is restricted to any family of tax schedules completely ordered by increasing progressivity, and that Hierarchical Adherence holds. Then the majority preference relation over the tax schedules is quasitransitive.

A graphical intuition for the result can be discerned from Fig. 1. In the figure we have depicted the net income schedules associated with two tax policies, where \( \tau_1 > \tau_0 \), so that \( \tau_1 \) is the less progressive schedule. Suppose that an individual of ability \( \theta_0 \) prefers the more progressive schedule, given his or her optimal labour supply, which is depicted as a point of tangency between an indifference curve \( U_0 \) and the \( \tau_0 \) schedule. Given Hierarchical
Adherence, any individual of ability $\theta_1 > \theta_0$ has 'flatter' indifference curves through every point in $(c, y)$ space. Hence he or she must prefer to supply at least as much labour as an individual of ability $\theta_0$. But, since tax schedules are single-crossing, this greater labour supply must be associated with a most-preferred tax schedule which is no more progressive than $\tau_0$. This is depicted in Fig. 1 as a point of tangency between an indifference curve $U_1$ and the less progressive schedule $\tau_1$. Hence individual preferences are single-crossing in $(\tau, \theta)$ given the optimal labour supply decision.

4. Application II: Objectives of labour unions

Efforts to understand equilibrium employment and wage determination in labour markets characterized by collective bargaining have been impeded by the difficulties in specifying the objectives of those representing workers in the bargaining process. Given that union leaderships are usually elected democratically by rank-and-file members, a natural and much-employed hypothesis has been that objectives are determined by direct votes of workers over wage–employment combinations that are feasible given the constraints imposed by the employer’s profit function and by the bargaining process.

The issue of existence of majority-voting equilibrium and the implied properties of union objectives was first examined rigorously by Blair and Crawford (1984). They considered the case of a ‘monopoly’ union, which sets the wage unilaterally, subject to the firm’s employment demand function, in order to maximize an objective function. Union members are expected utility maximizers. The firm is a closed shop, so that each worker’s
probability of being employed is a stochastic function only of the firm's aggregate level of employment (which is in turn a function of the wage set by the union) and the worker's own seniority within the union. Consequently, workers' objectives over wage–employment pairs are divergent.

Blair and Crawford show that if the probability distribution of employment demand shocks has an increasing hazard rate and if the firm's employment demand function is concave then worker's preferences over wage rates are single-peaked. Hence the majority preference relation is quasitransitive. They also observe, however, that inasmuch as the construction of the majority preference relation is completely ordinal in nature, there is no guarantee that it have a cardinal representation. Consequently, it cannot be shown that the union is an expected-utility maximizer, even if all its members are.

In this section we adopt the formulation of Blair and Crawford. We show that, if workers have homogeneous risk preferences, then monotonicity of the hazard rate is sufficient for the majority preference relation to be quasitransitive. Thus the assumption of a concave employment demand function is not required. Moreover, we show that Corollary 2 implies that the union is indeed an expected-utility maximizer.

Given that a worker is employed, we suppose that he or she works a fixed number of hours. Workers are identical in every respect but their seniority in the firm's hiring queue, so that the indirect utility of particular wage rate \( w \) is \( u(w) \).

For a given probability of unemployment \( p \), a worker's expected utility is given by

\[
(1 - p)u(w) + pu(B),
\]

where \( B \) is an unemployment benefit or alternative wage available to the worker with certainty in the event he or she not be hired by the firm.

Union members vote over the wage rate that will be set unilaterally by the union, taking the employment-demand behaviour of the firm as given. Employment demand is a stochastic function,

\[
\tilde{L} = L + \epsilon,
\]

where the stochastic component \( \epsilon \) is a pure expectational error with cumulative distribution function \( F(\epsilon) \). The non-stochastic component \( L \) is a known function of wages,

\[
L = \phi(w).
\]

We place no restrictions on the nature of the employment-demand function \( \phi \).

Suppose that the union and the firm impose a strict seniority rule in hiring and layoff decisions, and that \( \theta \) represents the worker's seniority; namely if
two workers are of types $\theta$ and $\theta'$, with $\theta < \theta'$, then when $\theta'$ is employed $\theta$ must also be. (Note this means that higher $\theta$ indicates lower seniority.) A worker $\theta$ is employed if and only if $L \geq \theta$, or equivalently if and only if $\epsilon \geq \theta - L$. The worker's expected utility from wage-employment combinations $(w, L)$ is therefore

$$U(w, L, \theta) = (1 - F(\theta - L))u(w) + F(\theta - L)u(B).$$

Union members then vote over employment-wage combinations that satisfy $L = \phi(w)$ in order to maximize $U(w, L, \theta)$. By Lemma 2, $U(w, \phi(w), \theta)$ is single-crossing in $(w, \theta)$ for all demand functions $\phi$ if and only if the marginal rate of substitution between employment and wages,

$$\sigma(w, L, \theta) = \frac{U(w, L, \theta)}{U_L(w, L, \theta)} = \frac{1}{f(\theta - L)} \frac{u(w)}{u(w) - u(B)},$$

is increasing or decreasing in $\theta$ for all $(w, L)$. Thus single-crossing obtains if and only if the distribution of employment shocks $\epsilon$ has a monotone increasing or decreasing hazard rate.

**Proposition 3.** In the union voting model of Blair and Crawford the majority preference relation is quasitransitive if the distribution of employment-demand shocks has a monotone hazard rate.

Note that our result imposes no restriction on the nature of the firm's employment demand (other than that it be a function) but we require that workers have identical risk preferences. In contrast, in order to derive single-peaked preferences over wages, Blair and Crawford require that the employment demand function be concave but impose no restrictions on preferences other than concavity.

In addition to providing a different sufficient condition for transitivity, single-crossing has a stronger implication than single-peakedness, which is of some significance in this application. Corollary 2 to Theorem 2 states that, when preferences are strictly single-crossing and the number of voters is odd, the majority preference relation coincides with the preferences of the median voter, and so inherits all its properties. This yields the following result, which need not obtain under single-peakedness.

**Proposition 4.** Suppose that the conditions of the previous proposition hold with strict monotonicity of the hazard rate. Suppose, further, that there is a unique median voter. Then the majority preference relation can be represented by a cardinally measured von Neumann–Morgenstern utility function.
A limitation of the model is our assumption that all workers have identical risk preferences. When this assumption is relaxed, the analysis becomes more complicated. It is not clear a priori how a worker's seniority within the union should be related to his or her attitude to risk and, consequently, it is difficult to determine how heterogeneity in risk preferences should influence workers' marginal rate of substitution between employment and wages, as expressed in (7).

5. Conclusion

We have demonstrated that an ordinal notion of the single-crossing condition, familiar in applications in public finance and information economics, is sufficient to guarantee the existence of a majority voting equilibrium. In so doing we unified in a simple and intuitive manner the domain restrictions stated in various forms by Roberts (1977), Epple and Romer (1991) and Rothstein (1990, 1991), among others. We have, in addition, provided a new set of techniques for verifying that the conditions obtain in applications. Our results also suggest that restrictive curvature conditions often imposed to obtain single-peaked preferences—in particular, the convexity of constraint sets of voting alternatives—may often be unnecessary to establish existence of voting equilibrium.

But existence was only the first step in developing a useful predictive theory of voting outcomes. In applications of the theory, one would also like to be able to characterize equilibrium outcomes and their comparative statics with respect to parameter changes. Comparative statics calculations have traditionally been difficult, since under the assumption of single-peakedness it is only possible to characterize the set of majority winners. When exogenous parameters change, the identity of the median voter may change as well, so that equilibrium outcomes change in unpredictable ways. Indeed, perhaps for this reason, comparative static exercises conducted for single-peaked preferences generally examine only parameter changes for which voters unanimously agree on the desired direction of change in the level of the public good. Under strict single-crossing, in contrast, a characterization of the entire majority preference relation is available, and comparative statics results follow easily, even when voters disagree about the desired direction of change.

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Appendix

Proof of Theorem 1. (i) If $x > y$, then $x \preceq \theta y \Rightarrow x \preceq \theta y$ for all $\theta > \theta_2$. Since $P$ is asymmetric, it follows that $\mu(x, y) \geq \mu(y, x)$. If $y > x$, then $x \preceq \theta y \Rightarrow x \preceq \theta y$ for all $\theta < \theta_1$, by the contrapositive of SC, and $x \preceq y$ once again.

(ii) Applying the previous argument with $x \preceq y$ and $x \preceq y$ yields $\mu(x, y) > \mu(y, x)$. Q.E.D.

Proof of Corollary 1. For any triple $x, y, z \in X$, suppose that $x \preceq y \preceq z$. By Theorem 1, this implies that $x \preceq y \preceq z$ and $x \preceq y \preceq z$. If any one of these four preferences is strict, then $x \preceq y \preceq z$ by the theorem.

Suppose therefore that $xI_1yI_1z$ and $xI_2yI_2z$. Then there are four possible cases to consider, corresponding to the following orders on $X$: (i) $x > y > z$; (ii) $x > z > y$; (iii) $z > y > x$; and (iv) $y > z > x$. The other orders on $X$ are not consistent with $x \preceq y \preceq z$ under SC. Cases (i) and (ii) are considered here; (iii) and (iv) are analogous, since they are merely the reverse orders.

(i) $x > y > z$: Given SC and the indifference of the median voters,

\[ x \preceq y \] or \[ y \preceq z \] or \[ x \preceq z \] \Rightarrow \theta > \theta_2, \quad (A.1)\]

\[ y \preceq x \] or \[ z \preceq y \] or \[ z \preceq x \] \Rightarrow \theta < \theta_1 \quad (A.2)\]

We can establish the following two implications as well.

(a) \[ z \preceq x \Rightarrow \mu(x, y) \preceq \mu(y, z) \]: \[ z \preceq x \] implies \[ \theta < \theta_1 \] by (A.2), which implies \[ y \preceq x \] or \[ y \preceq x \]. If \[ y \preceq x \] then \[ z \preceq x \] by transitivity.

(b) \[ x \preceq y \] or \[ y \preceq z \] \Rightarrow \mu(x, z) \preceq \mu(y, z) \]: \[ x \preceq y \] or \[ y \preceq z \] implies \[ \theta > \theta_2 \] by (A.1), which implies \[ x \preceq y \] and \[ y \preceq z \] and \[ z \preceq x \]. Hence \[ x \preceq z \] by transitivity.

Notice moreover that the sets of voters in $\Theta$ who strictly prefer $z$ to $y$ and those who prefer $y$ to $x$ must be nested. Thus (a) and (b) imply

\[ \mu(x, z) \geq \max\{\mu(x, y), \mu(y, z)\} \]

\[ \geq \mu(y, x) \]

and so $x \preceq y \preceq z$.

(ii) $x > z > y$: With this order, it follows from SC and the indifference of the median voters that

\[ x \preceq \theta y \] or \[ z \preceq y \] or \[ x \preceq y \] \Rightarrow \theta > \theta_2, \quad (A.1') \]
\( yP_\theta x \text{ or } yP_\theta z \text{ or } zP_\theta x \Rightarrow \theta < \theta_1 \). \[(A.2')\]

(a) \( zP_\theta x \Rightarrow yP_\theta x \): if not, then there exists a \( \theta \) such that \( zP_\theta x \) and \( xR_\theta y \).
Since \( \theta < \theta_1 \) by \((A.2')\), it must be that \( xI_\theta y \), so that \( zP_\theta y \), implying \( \theta > \theta_2 \), a contradiction.

(b) \( xP_\theta y \Rightarrow xP_\theta z \): \( xP_\theta y \) implies \( \theta > \theta_2 \) by \((A.1')\), which implies \( yR_\theta z \), which in turn implies \( xP_\theta z \).

These conditions taken together imply that \( \mu(x, z) \geq \mu(x, y) \geq \mu(y, x) \geq \mu(z, x) \), so that \( xP^Mz \). Q.E.D.

Proof of Theorem 2. Suppose there are two median voters. That \( xP_1y \) or \( xP_2y \Rightarrow xR^M_y \) was proven in Theorem 1. To prove the converse, suppose \( xR_\theta y \) and \( xR_\theta y \). If \( x > y \), then \( xR_\theta y \) implies \( xP_\theta y \) for all \( \theta > \theta_1 \), so \( \mu(x, y) > \mu(y, x) \) and \( xP^M_y \). If \( x < y \), then \( xR_\theta y \) implies \( xP_\theta y \) for all \( \theta < \theta_2 \) and the same result holds.

If there is one distinct median voter, suppose that \( x > y \). Then \( xR_\theta y \) implies \( \mu(x, y) \geq \mu(y, x) \) and hence \( xR^M_y \) by the foregoing argument, and \( xP_\theta y \) implies \( \mu(x, y) > \mu(y, x) \) so that \( xP^M_y \). The same arguments can be applied, reversing the order on \( \Theta \), when \( y > x \). Q.E.D.

Proof of Theorem 3. (i) \( SC \Rightarrow OR \): Choose any \( x, y \in X \) such that \( x > y \). By \( SC \), \( xP_\theta y \Rightarrow xP_\theta y \) for all \( \theta' > \theta \) and \( yP_\theta x \Rightarrow yP_\theta x \) for all \( \theta' < \theta \). Since \( R_\theta \) is complete, \((OR-1)\) obtains. When \( x < y \), \((OR-2)\) obtains by the analogous argument, reversing the order on \( \Theta \).

(ii) \( OR \Rightarrow SC \): Define an order \( \geq \) on \( X \) by: for all \( x, y \in X \), \( x \geq y \) if and only if \((OR-1)\) holds. Note that, since \((OR-1)\) or \((OR-2)\) holds for all \( x, y \in X \) by the hypothesis, \( \geq \) is complete and defines a chain on \( X \). Now choose any \( x, y \in X \) such that \( x \geq y \). With this definition, \((OR-1)\) implies that, for all \( \theta' > \theta \), \( xP_\theta y \Rightarrow xP_\theta y \) and \( xR_\theta y \) and \( xR_\theta y \Rightarrow xR_\theta y \). The two conditions together are \( SC \). Q.E.D.

Proof of Lemma 3. By Lemma 1, it suffices to show that, for all \( T \subset X \), \( \arg \max_{x \in T} v(x, \theta) \) is increasing in \( \theta \).

(a) Observe
\[
\arg \max_{x \in T} v(x, \theta) = \arg \max_{x \in T} \max_{y \in Y} u(x, y, \theta) = \Pi_X\left( \arg \max_{(x, y) \in T \times Y} u(x, y, \theta) \right),
\]
where \( \Pi_X \) is the projection operator onto \( X \). Note \( T \times Y \) is a sublattice of \( X \times Y \). Hence \( \arg \max_{(x, y) \in T \times Y} \) is increasing in \( \theta \) by Lemma 1. Since \( X \times Y \) has the product order, it follows that the projection onto \( X \) is increasing in the order on \( X \). Q.E.D.
(b) Define
\[
\phi(y) = \max_{x \in T} h(x, y),
\]
\[
\hat{x}(y) = \arg \max_{x \in T} h(x, y),
\]
and define
\[
y^*(\theta) = \arg \max_{y \in Y} u(\phi(y), y, \theta),
\]
where \(Y\) is the set of feasible private goods \(y\), which may depend on \(\theta\).

Given the separability structure of \(u\),
\[
x^*(\theta) = \arg \max_{x \in T} h(x, y^*(\theta))
\]
\[
= \hat{x}(y^*(\theta)).
\]

By Lemma 2, \(y^*\) is increasing in \(\theta\), since the Spence–Mirrlees condition holds. By Lemma 1, \(\hat{x}\) is increasing in \(y\), since \(h\) is single-crossing in \((x, y)\).

Hence \(x^*\) is increasing in \(\theta\). Q.E.D.

References

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