Econometrica, Vol. 49, No. 5 (September, 1981)

THE MEASUREMENT OF DEADWEIGHT LOSS REVISITED[1]

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The paper studies an economy with H households, $N + 1$ commodities, and M fixed factors with commodity taxes and government expenditures on goods and services. The first and second order directional derivatives of a certain weighted sum of utility functions with respect to any direction of tax change are calculated. The resulting measure of deadweight loss, due essentially to Boiteux, is contrasted with a measure based on Debreu's coefficient of resource utilization as well as with the more familiar measure of loss due to Hotelling and Harberger. The basic analytical technique used is the usual comparative statistics analysis, except that duality theory is used to simplify the computations.

1. INTRODUCTION

CONSIDER AN (ATEMPORAL) ECONOMY with H households, $N + 1$ variable commodities (such as various consumer goods and labor), M fixed factors (such as land, natural resources and various types of fixed capital), and a government which taxes commodities and fixed factors in order to finance various government expenditures. It is well known[2] that if the government can raise its required revenue by taxing the fixed factors alone, then the resulting allocation of resources is Pareto optimal—no single household's utility or real income can be increased without decreasing the utility of some other household.

Suppose we are at an initial equilibrium where government revenue is being raised by taxing the fixed factors alone. Then the resulting equilibrium can be rationalized by maximizing a certain weighted sum of utility functions subject to various feasibility constraints. Now think of the government replacing the taxes on fixed factors with distortionary commodity taxes. In Section 3, we calculate the second order directional derivative of the above weighted sum of utility functions with respect to any feasible direction of tax change, evaluated at the initial equilibrium which is Pareto optimal. Of course, the first order directional derivatives of the weighted sum of utility functions with respect to feasible directions of tax change are zero evaluated at this initial equilibrium. We obtain a measure of economic loss due to tax distortions which is virtually identical to that of Boiteux [**3**, p. 113] and which bears a resemblance to the "dead loss" of Hotelling [**22**, p. 254], the "consumer's surplus" measures of Hicks [**19**; **20**, pp. 330–3], and the "deadweight loss" measure of Harberger [**16**, p. 61; **17**, p. 788].

In Section 4, we calculate a measure of welfare loss based on Debreu's [**4**, **5**] coefficient of resource utilization (which is a modification of a measure of loss due to Allais [**1**, **2**]) and we show that under certain conditions, the Hotelling,

[1] This work was supported by National Science Foundation Grant SOC77-11105 at the Institute for Mathematical Studies in the Social Sciences, Stanford University and by a Canada Council grant. The author thanks C. Blackorby, M. Boiteux, A. Dixit, E. Sheshinsky, J. Weymark, and J. Whalley for helpful comments.

[2] See Hotelling [**22**], Stiglitz and Dasgupta [**31**, p. 155], or Diewert [**10**, p. 156].

Debreu, and Boiteux measure of welfare loss coincide. In this section we also introduce the concepts of *Debreu optimality* (which is analogous to Pareto optimality) and an *endowment reducing tax change* (which is analogous to the concept of a strict Pareto improving tax change studied recently by Hahn [**15**], Guesnerie [**14**], Harris [**18**], Dixit [**12**], and Weymark [**32**, **33**]), and we prove some propositions involving these concepts.

Our model of the economy is explained in the following section. It is quite similar to the Diamond-Mirrlees [**7**] model with three major differences: (i) we assume that there are fixed inelastically supplied primary factors of production in the economy, (ii) we use expenditure functions instead of indirect utility functions in order to describe consumer's preferences, and (iii) we use a variable profit function instead of a transformation function in order to describe technology.

2. THE MODEL

Our model is explained in more detail in Diewert [**10**]; however, in the present paper, specific taxes replace the ad valorem taxes used in Diewert [**10**]. We assume that the number of finally demanded goods, intermediate goods, and types of labor is $N+1$ (one of these goods will play the role of a numeraire good) and that there are an additional M fixed factors ("land" and types of "capital") in the economy.

Denote the economy's aggregate production possibilities set as Y and let $v \gg 0_M$[3] denote the M dimensional vector of fixed factors which is available to the economy.

Define the *variable profit function* π corresponding to Y as the solution to the following profit maximization problem:

$$\pi(p,v) = \pi(p_0, p_{.}, v) \equiv \max_y\{p^T y : (y,v) \in Y\} \tag{1}$$

where $p_0 > 0$ is the price of the 0th finally demanded good, $p_{.} \equiv (p_1, \ldots, p_N)^T \gg 0_N$ is an N dimensional vector whose components are the positive prices of other finally demanded (and supplied) goods which producers face, $p^T \equiv (p_0, p_{.}^T)$ and $y^T \equiv (y_0, y_{.}^T) \equiv (y_0, y_1, y_2, \ldots, y_N)$ is a vector of net outputs which the aggregate production sector can supply, given that it can utilize the vector v of fixed inputs. If $y_n > 0$, then the nth good is supplied by the production sector; if $y_n < 0$, then the nth good is an input into the production sector. By Hotelling's Lemma [**21**, p. 594], $\partial\pi(p,v)/\partial p_n = y_n(p,v)$ for $n = 0, 1, \ldots, N$ where $y_n(p,v)$ is the (variable) profit maximizing amount of output n (of input n if $y_n(p,v) < 0$) given variable good prices p and the vector of fixed

[3]Notation: $v \gg 0_M$ means that each component of the M-dimensional vector v is positive; $v \geqq 0_M$ means each component of v is nonnegative; $v > 0_M$ means $v \geqq 0_M$ but $v \neq 0_M$ where 0_M is a vector of zeroes; $r^T v \equiv \sum_{m=1}^{M} r_m v_m$ denotes the inner product of r and v.

inputs v, i.e.,

$$(2) \qquad y = y(p,v) \equiv \nabla_p \pi(p,v)$$

where $\nabla_p \pi(p,v)$ denotes the column vector of first order partial derivatives of π with respect to the components of p.

It can also be shown (Diewert [**8**, p. 140], Lau [**23**]) that if the variable profit function $\pi(p,v)$ is differentiable with respect to the components of the vector of fixed inputs v, then the vector of competitive returns to the fixed inputs r is equal to:

$$(3) \qquad r = r(p,v) \equiv \nabla_v \pi(p,v) > 0_M$$

where $\nabla_v \pi(p,v) \equiv [\partial \pi(p,v_1,\ldots,v_M)/\partial v_1,\ldots,\partial \pi(p,v_1,\ldots,v_M)/\partial v_M]^T$ is the (column) vector of partial derivatives of π with respect to the components of v.

Using Euler's Theorem on homogeneous functions, it can be shown that

$$(4) \qquad p^T \nabla_p \pi(p,v) \equiv p_0 y_0 + p_{\cdot}^T y_{\cdot} = \pi(p,v)$$

and

$$(5) \qquad \nabla_{pp}^2 \pi(p,v) p \equiv \begin{bmatrix} \pi_{00}, & \pi_{0\cdot} \\ \pi_{\cdot 0}, & \pi_{\cdot\cdot} \end{bmatrix} \begin{bmatrix} p_0 \\ p_{\cdot} \end{bmatrix} = \begin{bmatrix} 0 \\ 0_N \end{bmatrix}$$

where $\nabla_{pp}^2 \pi$ denotes the matrix of second order partial derivatives of π with respect to p.

The hth household's *expenditure function* m^h is defined as the minimum net cost (labor supplies are indexed negatively) of achieving a given utility level; i.e., $m^h(u^h,q) \equiv \min_x \{q^T x : f^h(x) \geqq u^h\}$ where $(q_0,q_1,\ldots,q_N) \equiv q^T \gg 0_{N+1}^T$ are the positive prices consumers face, f^h is the hth household's utility function, and u^h belongs to the range of f^h.

If m^h is differentiable with respect to the components of the price vector $q \equiv (q_0,q_1,\ldots,q_N) \equiv (q_0,q_{\cdot})$, then it can be shown (Hicks [**20**, p. 331], Shephard [**27**, p. 11], Diamond and McFadden [**6**, p. 4]) that the expenditure minimizing demand for the ith good needed to achieve utility level u^h is $x_i^h(u^h,q) = \partial m^h(u^h,q)/\partial q_i$, $i = 0,1,2,\ldots,H$:

$$(6) \qquad \begin{aligned} x_0^h &\equiv x_0^h(u^h,q) = \nabla_{q_0} m^h(u^h,q) \equiv \partial m^h(u^h,q)/\partial q_0, \\ x_{\cdot}^h &\equiv x_{\cdot}^h(u^h,q) = \nabla_{q_{\cdot}} m^h(u^h,q), \quad h = 1,2,\ldots,H, \end{aligned}$$

where $x_{\cdot}^h \equiv (x_1^h,\ldots,x_N^h)^T$ and $\nabla_{q_{\cdot}} m^h(u^h,q)$ is the vector of partial derivatives of m^h with respect to the N components of $q_{\cdot} \equiv (q_1,q_2,\ldots,q_N)^T$. Linear homogeneity of m^h in q implies

$$(7) \qquad \begin{bmatrix} \nabla_{q_0 q_0}^2 m^h, & \nabla_{q_0 q_{\cdot}}^2 m^h \\ \nabla_{q_{\cdot} q_0}^2 m^h, & \nabla_{q_{\cdot} q_{\cdot}}^2 m^h \end{bmatrix} q \equiv \begin{bmatrix} \sigma_{00}^h, & \sigma_{0\cdot}^h \\ \sigma_{\cdot 0}^h, & \sigma_{\cdot\cdot}^h \end{bmatrix} \begin{bmatrix} q_0 \\ q_{\cdot} \end{bmatrix} = \begin{bmatrix} 0 \\ 0_N \end{bmatrix}, \quad h = 1,2,\ldots,H.$$

The aggregate consumer substitution (or Slutsky) matrix,

$$(8) \qquad \sigma \equiv \begin{bmatrix} \sigma_{00}, & \sigma_{0\cdot} \\ \sigma_{\cdot 0} & \sigma_{\cdot\cdot} \end{bmatrix} \equiv \sum_{h=1}^{H} \begin{bmatrix} \sigma_{00}^h & \sigma_{0\cdot}^h \\ \sigma_{\cdot 0}^h & \sigma_{\cdot\cdot}^h \end{bmatrix},$$

is a negative semidefinite symmetric matrix which satisfies the restrictions (7).

We also make the following *local* assumptions on the derivatives of the expenditure functions with respect to the utility levels:

$$(9) \qquad \frac{\partial m^h}{\partial u^h}(u^h, q) = 1, \quad \frac{\partial^2 m^h}{\partial u^h \partial u^h}(u^h, q) = 0, \quad h = 1, \ldots, H.$$

The above restrictions simply provide convenient local cardinalizations of utility.[4] Some further implications of (9) can be obtained after we introduce some additional notation. Define for $h = 1, 2, \ldots, H$:

$$(10) \qquad \begin{aligned} \eta_0^h &\equiv \frac{\partial x_0^h(u^h, q)}{\partial u^h} = \frac{\partial^2 m^h(u^h, q)}{\partial q_0 \partial u^h}, \\ \eta_{\cdot}^h &\equiv \frac{\partial x_{\cdot}^h(u^h, q)}{\partial u^h} = \nabla_{q.u^h}^2 m^h(u^h, q), \end{aligned}$$

where the equalities in (10) follow from Shephard's Lemma (6), and where $\eta_{\cdot}^h \equiv [\eta_1^h, \ldots, \eta_N^h]^T$. The η_i^h can be interpreted as ordinary income derivatives. Moreover, since $\partial m^h(u^h, q)/\partial u^h$ is linearly homogeneous in q, Euler's theorem on homogeneous function plus the normalizations in (9) imply that the η_i^h satisfy the following additional restrictions:

$$(11) \qquad 1 = \frac{\partial m^h}{\partial u^h}(u^h, q) = \sum_{i=0}^{N} q_i \frac{\partial^2 m^h}{\partial u^h \partial q_i}(u^h, q) \equiv q_0 \eta_0^h + q_{\cdot}^T \eta_{\cdot}^h, \quad h = 1, \ldots, H.$$

We assume that the hth household owns the resource vector $v^h \geqq 0_M$ and that

$$(12) \qquad \sum_{h=1}^{H} v^h = v \gg 0_M$$

where v is the vector of fixed resources which appeared as an input into the aggregate production sector.

We assume that the government has the power to tax commodities and fixed factors. We assume that there are no tax distortions within the production sector, but there are tax wedges between the prices producers face (p, r) and the prices consumers face (q, s). Specifically, assume that $q_i = p_i + t_i$ for $i = 0, 1, 2, \ldots, N$ where $p_i > 0$ is the ith producer price and t_i is the tax on the ith good. If the ith

[4] They imply that (local) changes in real income are proportional to changes in nominal income when prices are held fixed at the initial equilibrium prices q. This type of scaling convention is consistent with the (global) scaling conventions suggested by Samuelson [**26**, p. 1262] and Hicks [**19**].

good is being supplied by the production sector, then $t_i > 0$ ($t_i < 0$) implies that the ith good is being taxed (subsidized) by the government, but if the ith good is being demanded by the aggregate production sector, then $t_i < 0$ ($t_i > 0$) implies that the ith commodity, a type of labor service, is being taxed (subsidized) by the government. The relationship between producer and consumer prices can be summarized as:

$$(13) \qquad q = p + t \quad \text{or} \quad \begin{aligned} q_0 &= p_0 + t_0, \\ q_{\cdot} &= p_{\cdot} + t_{\cdot}. \end{aligned}$$

Initially, one unit of the mth fixed factor earns a reward $r_m \geqq 0$, but the government taxes this reward at the rate τ_m (subsidizes if $\tau_m < 0$) so that the after tax return to households is $s_m \equiv r_m - \tau_m$. Again we can summarize the relationship between producer and consumer prices using vector notation as:

$$(14) \qquad s \equiv r - \tau.$$

We assume that the government spends its tax revenue on purchases of goods and services $(x_0^0, x_1^0, \ldots, x_N^0) \equiv (x_0^0, x_{\cdot}^{0T})$ in order to produce "government services" or "public goods," which are not listed as arguments in the household utility functions or the private production functions, since we hold x_0^0 and $x_{\cdot}^0$ constant, in order to simplify our derivations. We assume that $x_i^0 \geqq 0$ whether the ith commodity is being supplied or demanded by the aggregate private production sector for $i = 0, 1 \ldots N$, with $x_i^0 > 0$ for at least one index i.

We can now list the basic demand equals supply equations in our model.

Household expenditure is equal to after tax income from fixed factors for each household and thus using equation (3):

$$(15) \qquad m^h(u^h, p_0 + t_0, p_{\cdot} + t_{\cdot}) = v^{hT}(\nabla_v \pi(p_0, p_{\cdot}, v) - \tau), \quad h = 1, 2, \ldots, H,$$

or $q^T x^h = v^{hT} s$ for $h = 1, 2, \ldots, H$ using equations (13) and (14). We also have consumer plus government net demand equals private producer net supply for goods 0 and 1 to N, and thus using equations (2) and (6),

$$(16) \qquad \sum_{h=1}^{H} \nabla_{q_0} m^h(u^h, p_0 + t_0, p_{\cdot} + t_{\cdot}) + x_0^0 = \nabla_{p_0} \pi(p_0, p_{\cdot}, v),$$

$$(17) \qquad \sum_{h=1}^{H} \nabla_{q_{\cdot}} m^h(u^h, p_0 + t_0, p_{\cdot} + t_{\cdot}) + x_{\cdot}^0 = \nabla_{p_{\cdot}} \pi(p_0, p_{\cdot}, v).$$

The above equations can be rewritten as $\sum_{h=1}^{H} x^h + x^0 = y$. Finally, the government's budget constraint can be written as revenue from taxing goods 0 to N plus revenue from taxing fixed factors equals government expenditures on goods and services valued at consumer prices:

$$(18) \qquad t_0 \nabla_{p_0} \pi(p_0, p_{\cdot}, v) + t_{\cdot}^T \nabla_{p_{\cdot}} \pi(p_0, p_{\cdot}, v) + \tau^T \nabla_v \pi(p_0, p_{\cdot}, v)$$
$$= (p_0 + t_0) x_0^0 + (p_{\cdot} + t_{\cdot})^T x_{\cdot}^0.$$

Equation (24) may be rewritten in a more readily understandable form as $t^T y + \tau^T r = q^T x^0$.

Equations (15) to (18) are the basic equilibrium relations of our model. It can be shown that not all of the $H + N + 2$ equations are independent: any one of the equations can be derived from the other equations. We shall choose to drop the government budget constraint. Moreover (see Munk **[25]**), there is no loss of generality in fixing one of the producer prices. In what follows, we take the producer price $p_0 > 0$ to be our numeraire and express all other prices relative to p_0.

Thus we are left with $H + N + 1$ independent equations in $H + 2N + 1 + M$ unknowns. We assume that the H utility levels $u \equiv (u^1, u^2, \ldots, u^H)^T$, the N producer prices $p_. \equiv (p_1, p_2, \ldots, p_N)^T$, and *one* of the tax rates can be determined implicitly as functions of the remaining $N + M$ tax rates, at least locally around the initial equilibrium.

Totally differentiate equations (15), (16), and (17) and we obtain the following system of equations which is approximately valid for small changes in the $M + N$ exogenous tax variables (one tax rate is endogenous):

$$A\Delta u = B_0 \Delta p_. + B_1 \Delta t_0 + B_2 \Delta t_. + B_3 \Delta t \tag{19}$$

where

$$A \equiv \begin{bmatrix} 1, & 0, \ldots, & 0 \\ \vdots & & \vdots \\ 0, & 0, \ldots, & 1 \\ \eta_0^1, & \eta_0^2, \ldots, & \eta_0^H \\ \eta_.^1, & \eta_.^2, \ldots, & \eta_.^H \end{bmatrix}, \quad B_0 \equiv \begin{bmatrix} -x_.^{1T} + v^{1T}\nabla_{vp.}^2 \pi \\ \vdots \\ -x_.^{HT} + v^{HT}\nabla_{vp.}^2 \pi \\ -\sigma_{0.} + \pi_{0.} \\ -\sigma_{..} + \pi_{..} \end{bmatrix},$$

$$B_1 \equiv \begin{bmatrix} -x_0^1 \\ \vdots \\ -x_0^H \\ -\sigma_{00} \\ -\sigma_{.0} \end{bmatrix}, \quad B_2 \equiv \begin{bmatrix} -x_.^{1T} \\ \vdots \\ -x_.^{HT} \\ -\sigma_{0.} \\ -\sigma_{..} \end{bmatrix}, \quad B_3 \equiv \begin{bmatrix} -v^{1T} \\ \vdots \\ -v^{HT} \\ 0_M^T \\ 0_{N\times M} \end{bmatrix},$$

and $\Delta u \equiv [\Delta u^1, \Delta u^2, \ldots, \Delta u^H]^T$ is a vector of changes in utility levels, $\Delta p_. \equiv [\Delta p_1, \Delta p_2, \ldots, \Delta p_N]^T$ is a vector of changes in the N variable commodity prices (recall that $\Delta p_0 \equiv 0$ since commodity 0 is the numeraire good), Δt_0 is the change in the tax rate for commodity 0, $\Delta t_. \equiv [\Delta t_1, \ldots, \Delta t_N]^T$ is the vector of tax changes for (variable) commodities 1 to N, $\Delta\tau \equiv [\Delta\tau_1, \ldots, \Delta\tau_M]^T$ is the vector of tax changes for the fixed factors, and the other variables and parameters have been defined above.

The government's *optimal taxation problem* can be phrased as: maximize $F(u^1, u^2, \ldots, u^H)$ with respect to $u, p_., t_0, t_., \tau$ subject to the constraints (15), (16), and (17), where F is the government's (strictly increasing in the household utilities) social welfare function. Linearizing the objective function and the constraints of this constrained maximization problem around an initial equilibrium point satisfying the constraints (15)–(17) yields the following *optimal tax perturbation problem*:

$$\max_{\Delta u,\,\Delta p_.,\,\Delta t_0,\,\Delta t_.,\,\Delta\tau} \alpha^T\Delta u \quad \text{subject to (19)}$$

where $\alpha^T \equiv (\alpha_1, \alpha_2, \ldots, \alpha_H) \equiv \nabla_u^T F(u) > 0_H$ is the vector of first order partial derivatives of the social welfare function F evaluated at the initial equilibrium utility levels.

In general, the government will be able to choose only $N + M$ of the $1 + N + M$ tax rates in an independent manner; the remaining tax rate becomes an endogenous variable (recall that we are holding real government expenditures fixed).

For the sake of definiteness, we assume that t_0 is the endogenous tax rate and thus $t_.$ and τ are the vectors of independent tax rates. Our differentiability assumptions plus the Implicit Function Theorem imply the existence of functions $u(t_., \tau)$, $p_.(t_., \tau)$, and $t_0(t_., \tau)$ in a neighborhood of the original equilibrium tax rates such that equations (15)–(17) are satisfied, provided that the following assumption is satisfied:

(20) $\qquad [A, -B_0, -B_1]^{-1}$ exists.

Assuming (20), the *optimal tax perturbation problem* can be rewritten as the following unconstrained maximization problem in the independent government tax instruments $\Delta t_.$ and $\Delta\tau$:

$$(21) \qquad \max_{\Delta t_.,\,\Delta\tau} \alpha^T[I_H, 0_{H\times N}, 0_N][A, -B_0, -B_1]^{-1}[B_2, B_3]\begin{bmatrix}\Delta t_. \\ \Delta\tau\end{bmatrix}.$$

From (19), it can be seen that $[A, -B_0, -B_1]^{-1}[B_2, B_3]$ is the $H + N + 1$ by $N + M$ matrix of the first order partial derivatives of the implicit functions $u(t_., \tau)$, $p_.(t_., \tau)$, and $t_0(t_., \tau)$ with respect to the components of $t_.$ and τ. Thus the coefficient vector in (21) (call it $\beta^T \equiv [\beta_1, \ldots, \beta_N, \beta_{N+1}, \ldots, \beta_{N+M}]$) can be interpreted as the vector of partial derivatives of the government's social welfare function with respect to the independent tax instruments $t_1, \ldots, t_N$, $\tau_1, \ldots, \tau_M$, evaluated at our initial equilibrium point. Hence, in order to find the *optimal direction of tax change* (i.e., the directional derivative of the social welfare function with respect to the $N + M$ independent or exogenous tax parameters) which will lead to the greatest rate of increase in the objective function, we need to add the following normalization to (21):

$$(22) \qquad \sum_{n=1}^{N} (\Delta t_n)^2 + \sum_{m=1}^{M} (\Delta\tau_m)^2 = 1.$$

It is straightforward to verify that the *optimal tax perturbation* (i.e., the solution to (21) and (22) is $(\Delta t_.^T, \Delta\tau^T) \equiv \beta^T/(\beta^T\beta)^{1/2}$ provided $\beta \neq 0_{N+M}$ where β^T is the coefficient vector in (21).

On the other hand, if $\beta = 0_{N+M}$, then the partial derivatives of the objective function with respect to the independent tax rates $t_1, \ldots, t_N, \tau_1, \ldots, \tau_M$ are all zeroes; i.e., the *first order necessary conditions for the initial set of taxes to be α optimal* are satisfied.

It is possible to express these conditions for α optimality in some alternative ways which will prove to be convenient. Referring back to equation (19), it can be seen that we have the following *necessary condition for taxes to be α optimal*:

(23) if z^1, z^2 is a solution to $Az^1 = Bz^2$, then $\alpha^T z^1 = 0$,

where $z^1 \equiv \Delta u$, $z^{2T} \equiv [\Delta p_{\cdot}^T, \Delta t_0, \Delta t_{\cdot}^T, \Delta \tau^T]$, $B \equiv [B_0, B_1, B_2, B_3]$, and the matrices A, B_0, B_1, B_2, B_3 are defined below (19).

Using Motzkin's Theorem of the Alternative,[5] it can be verified that (23) is equivalent to the following *necessary condition for taxes to be α optimal*:

(24) $\exists w$ such that $w^T B = 0_{N+1+N+M}^T$ and $w^T A = \alpha^T$.

Now suppose that the required government revenue is being raised by taxing (and possibly subsidizing) only the fixed factors so that commodity taxes on variable goods and services are initially zero (i.e., $t_0 = 0$ and $t_{\cdot} = 0_N$). Then by (13), $q_0 = p_0$; $q_{\cdot} = p_{\cdot}$ and thus using (5), $q_0 \pi_{0\cdot} + q_{\cdot}^T \pi_{\cdot\cdot} = 0_N^T$. Using this last relation, (7) and (11), it can be verified that $w^T B = 0_{N+1+N+M}^T$ and $w^T A = 1_H^T$ where $w^T \equiv [0_H^T, q_0, q_{\cdot}^T]$ and 1_H is a vector of ones. Thus the necessary condition for the initial system of taxes to be α optimal (24) is satisfied for $\alpha \equiv 1_H$.

In the following section, we shall calculate the first and second order directional derivatives of the social welfare function $1_H^T u$ with respect to feasible directions of tax change, evaluated at an initial equilibrium point where the required revenue is being raised by taxing fixed factors alone.

3. BOITEUX'S MEASURE OF DEADWEIGHT LOSS

Suppose that the government can raise its required revenue by taxing fixed factors alone. Then as we have seen above, the necessary condition for α optimality is satisfied for $\alpha = 1_H$.[6] Let the initial vector of taxes on the fixed factors be denoted by τ^*, while the other tax rates are initially zero.

Assuming (20), the Implicit Function Theorem implies the existence of functions $u(t_{\cdot}, \tau)$, $p_{\cdot}(t_{\cdot}, \tau)$, and $t_0(t_{\cdot}, \tau)$ in a neighborhood of the initial equilibrium tax rates, $t_{\cdot} = 0_N$ and $\tau = \tau^*$, such that equations (15)–(18) are satisfied. Now pick a direction of tax change, i.e., let the independent tax rates $t_{\cdot}$ and τ be

[5] *Either* $Ex \gg \underline{0}$, $Fx \geqq \underline{0}$, $Gx = \underline{0}$ has a solution (where E is a nonvacuous matrix, F and G are matrices, and x is a vector of variables) *or* $y^{1T}E + y^{2T}F + y^{3T}G = \underline{0}^T$, $y^1 > \underline{0}$, $y^2 \geqq \underline{0}$ has a solution where y^1, y^2, and y^3 are vectors of variables. See Mangasarian [**24**, p. 34].

[6] Since the government can raise its required revenue by taxing fixed factors (some fixed factors could be subsidized), the initial allocation of resources is Pareto optimal and this allocation could be generated by maximizing a weighted sum of utility functions. The fact that the vector of utility weights α turns out to be a vector of ones is a consequence of our scaling of utility assumptions (9).

defined as the following functions of the scalar variable ξ:

(25) $$t_{.} \equiv \tilde{t}_{.}\,\xi \quad \text{and} \quad \tau \equiv \tau^{*} + \tilde{\tau}\xi$$

where $\tilde{t}_{.} \equiv (\tilde{t}_1, \ldots, \tilde{t}_N)^T$, $\tilde{\tau} \equiv (\tilde{\tau}_1, \ldots, \tilde{\tau}_M)^T$, and $\sum_{n=1}^{N}\tilde{t}_n^2 + \sum_{m=1}^{M}\tilde{\tau}_m^2 = 1$. Now all of the variables in the economy, u, $p_{.}$, t_0, $t_{.}$, and τ, can be regarded as functions of the scalar variable ξ. In this section, we will calculate the first and second derivatives of the social welfare function $\sum_{h=1}^{H}u^h(\xi)$, evaluated at the initial equilibrium point, $\xi = 0$.

Premultiply equation (16) by p_0, premultiply (17) by $p_{.}^T$, and add the resulting equations. Differentiate the resulting equation with respect to ξ, regarding $u, p_{.}$, t_0, $t_{.}$, and τ as functions of ξ. We find after using (16), (17), and cancelling some terms, that

(26) $$\sum_{h=1}^{H} \nabla_{u^h} m^h(u^h, p_0 + t_0, p_{.} + t_{.})\frac{\partial u^h}{\partial \xi} = \sum_{i=0}^{N}\sum_{h=1}^{H} t_i \frac{\partial}{\partial \xi}\left[\nabla_{q_i} m^h(u^h, p_0 + t_0, p_0 + t_{.})\right].$$

When $\xi = 0$, $t_0 = 0$ and $t_{.} = 0_N$ so that upon making use of the normalizations (9), (26) reduces to

(27) $$\sum_{h=1}^{H} 1\,\frac{\partial u^h}{\partial \xi} = 0.$$

Thus the first order condition for maximizing $\sum u^h(\xi)$ with respect to ξ is satisfied at $\xi = 0$ and, to the first order, welfare will remain unchanged with respect to variations in ξ for all feasible directions of tax change.

Now differentiate (26) with respect to ξ. Evaluating the derivatives at $\xi = 0$ (so that $t_0 = 0$ and $t_{.} = 0_N$ again) and making use of the normalizations (9), we find that the second order derivative of our social welfare function with respect to ξ is

(28) $$\frac{\partial^2}{\partial \xi^2}\left[\sum_{h=1}^{H} u^h\right] = -\sum_{i=0}^{N}\sum_{h=1}^{H}\left(\frac{\partial u^h}{\partial \xi}\right)\left(\nabla^2_{u^h q_i} m^h\right)\left(\frac{\partial p_i}{\partial \xi}\right) + \sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{\partial t_i}{\partial \xi}\right)\left(\sum_{h=1}^{H}\nabla^2_{q_i q_j} m^h\right)\left(\frac{\partial t_j}{\partial \xi} + \frac{\partial p_j}{\partial \xi}\right)$$

where the derivatives of m^h are evaluated at the initial equilibrium point $(u^{h*}, p_0^* + t_0^*, p_{.}^* + t_{.}^*$ where $t_0^* = 0$ and $t_{.}^* = 0_N)$. The partial derivatives of u, $p_{.}$, t_0, $t_{.}$, τ, and p_0 with respect to ξ are defined as:

(29) $$\partial p_0/\partial \xi \equiv 0, \quad \nabla_{\xi}\tau \equiv \tilde{\tau}, \quad \nabla_{\xi} t_{.} \equiv \tilde{t}_{.}, \quad \text{and}$$
$$\left[\nabla_{\xi}u^T, \nabla_{\xi}p_{.}^T, \nabla_{\xi}t_0\right]^T \equiv \left[A, -B_0, -B_1\right]^{-1}\left[B_2, B_3\right]\left[\tilde{t}_{.}^T, \tilde{\tau}^T\right]^T$$

where

$$\nabla_\xi u \equiv [\partial u^1/\partial\xi, \ldots, \partial u^H/\partial\xi]^T,$$

$$\nabla_\xi p. \equiv [\partial p_1/\partial\xi, \ldots, \partial p_N/\partial\xi]^T,$$

$$\nabla_\xi t_0 \equiv \partial t_0/\partial\xi, \tilde{t}. \equiv [\tilde{t}_1, \ldots, \tilde{t}_N]^T,$$

$$\tilde{\tau} \equiv [\tilde{\tau}_1, \ldots, \tilde{\tau}_M]^T,$$

and A, B_0, B_1, B_2, B_3 are defined below (19).

Differentiate the equilibrium equations (17) with respect to ξ, evaluating the derivatives at $\xi = 0$. Premultiply the resulting ith equation through by $\partial p_i/\partial\xi$ and sum the resulting equations. Remembering that $\partial p_0/\partial\xi \equiv 0$, we obtain the following equation where the partial derivatives of π are evaluated at the initial equilibrium point (p_0^*, p^*, v):

$$(30)\qquad \sum_{i=0}^{N}\sum_{h=1}^{H}\left(\frac{\partial p_i}{\partial\xi}\right)\left(\nabla^2_{q_i u^h} m^h\right)\left(\frac{\partial u^h}{\partial\xi}\right) + \sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{\partial p_i}{\partial\xi}\right)\left(\sum_{h=1}^{H}\nabla^2_{q_i q_j} m^h\right)\left(\frac{\partial p_j}{\partial\xi}+\frac{\partial t_j}{\partial\xi}\right) = \sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{\partial p_i}{\partial\xi}\right)\left(\nabla^2_{p_i p_j}\pi\right)\left(\frac{\partial p_j}{\partial\xi}\right).$$

Substitute (30) into (28) and obtain

$$(31)\qquad \frac{\partial^2}{\partial\xi^2}\left[\sum_{h=1}^{H} u^h\right] = \sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{\partial t_i}{\partial\xi}+\frac{\partial p_i}{\partial\xi}\right)\left(\sum_{h=1}^{H}\sigma_{ij}^h\right)\left(\frac{\partial t_j}{\partial\xi}+\frac{\partial p_j}{\partial\xi}\right) - \sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{\partial p_i}{\partial\xi}\right)(\pi_{ij})\left(\frac{\partial p_j}{\partial\xi}\right)$$

where $\sigma_{ij}^h \equiv \nabla^2_{q_i q_j} m^h$ is the hth consumer's compensated demand derivative for good i with respect to a change in q_j (recall (6)), $\pi_{ij} \equiv \partial^2\pi/\partial p_i \partial p_j$ is the derivative of the ith aggregate supply function with respect to a change in p_j (recall (2)) and the derivatives $\partial t_i/\partial\xi$ and $\partial p_j/\partial\xi$ are defined by (29). Since $\sigma \equiv \sum_{h=1}^{H}\sigma^h \equiv [\sum_{h=1}^{H}\sigma_{ij}^h]$ is a negative semidefinite symmetric $N+1$ by $N+1$ matrix and $[\pi_{ij}]$ is a positive semidefinite symmetric matrix, (31) shows that the second order derivative with respect to ξ of our social welfare function, $\sum_{h=1}^{H}u^h$, is nonpositive for all directions of tax change defined by $\tilde{t}.$ and $\tilde{\tau}$. Thus a movement away from the initial system of "pure" fixed factor taxes towards a system with distortionary commodity taxes will generally lead to a lower level of social welfare. If we take a second order Taylor Series expansion of our social welfare function $\sum u^h$ with respect to ξ, then using (27) and (31), we find that the negative of the welfare change is a nonnegative measure of the welfare loss due to a movement towards distortionary commodity taxation which for small ξ^* is approximately equal to the following measure of welfare loss,[7] due essentially to Boiteux [**3**, p. 122]:[8]

$$(32) \qquad L_B \equiv -\frac{1}{2}\xi^{*2}\partial^2\left[\sum_{h=1}^{H} u^h(\xi)\right]\Big/\partial\xi^2$$

where $\partial^2[\sum u^h]/\partial\xi$ is defined by (31).

Although we end up with the same formula as Boiteux, there are some differences in our underlying models: (i) Boiteux [**3**, p. 119] measures individual losses in terms of a Hicksian [**19**, **20**, pp. 330–3] consumer surplus concept instead of direct changes in utility;[9] (ii) Boiteux has no explicit fixed factors; (iii) Boiteux has no explicit budget constraints relating the profits of firms or returns to fixed factors to ownership claims of consumers; and (iv) Boiteux does not explicitly mention the constraints on the derivatives of the endogenous variables with respect to the exogenous tax variables, (29). Also, the following quotation by Boiteux [**3**, p. 118] indicates that he rejected the idea of measuring losses as a sum of utility changes:

> "En particulier la variation de la somme de satisfactions individuelles pondérées par les désirabilities marginales de la monnaie ne peut etre *a priori* considérée comme une expression de la perte. Il n'est pas certain, d'aileurs, qu'une telle expression soit définie positive."

We conclude this section by providing a further interpretation of the two terms on the right hand side of (31). From equation (2), the system of aggregate (net) supply functions is $y(p,v) \equiv \nabla_p \pi(p,v)$. Now regard y as a function of p and hence y is also a function of ξ, and thus the ith component of $y(p(\xi),v)$ has the following derivative with respect to ξ evaluated at $\xi = 0$:

$$(33) \qquad \frac{\partial y_i}{\partial \xi} \equiv \sum_{j=0}^{N} \pi_{ij}\left(\frac{\partial p_j}{\partial \xi}\right) \quad \text{for} \quad i = 0, 1, \ldots, N.$$

From (6), the hth consumer has the system of commodity demand (and labor supply) functions $x^h(u^h,q) \equiv \nabla_q m^h(u^h,q)$. Now regard u^h and $q_i \equiv p_i + t_i$ as functions of ξ and hence x^h is also a function of ξ and the ith component of $x^h(u^h(\xi),q(\xi))$ has the following derivative with respect to ξ evaluated at $\xi = 0$:

$$(34) \qquad \frac{\partial x_i^h}{\partial \xi} = \eta_i^h\left(\frac{\partial u^h}{\partial \xi}\right) + \sum_{j=0}^{N} \sigma_{ij}^h\left(\frac{\partial t_j}{\partial \xi} + \frac{\partial p_j}{\partial \xi}\right), \quad i = 0, 1, \ldots, N.$$

[7] Hotelling [**22**, p. 254] uses the term "dead loss" while Harberger [**16**, p. 58] uses the term "deadweight loss." For an exposition and additional references to the topic, see Harberger [**17**]. On the consistency of the Hotelling-Harberger loss measure with revealed preference theory, see Diewert [**9**].

[8] The equivalence becomes clear if we note that $\partial q_i/\partial\xi = \partial t_i/\partial\xi + \partial p_i/\partial\xi$ in (31). The loss measure is approximate because we are neglecting terms beyond the second order.

[9] However, our normalizations (9) lead to the equivalence of the two methods of measuring losses.

Define the hth consumer's constant utility system of commodity demand functions with respect to variations in ξ as $\bar{x}^h(\xi) \equiv \nabla_q m^h(u^h, q(\xi))$; i.e., u^h is to be held constant as ξ varies. The ith component of $\bar{x}^h(\xi)$ has the following derivative evaluated at $\xi = 0$:

$$(35) \qquad \frac{\partial \bar{x}_i^h}{\partial \xi} \equiv \sum_{j=0}^{N} \sigma_{ij}^h \left(\frac{\partial t_j}{\partial \xi} + \frac{\partial p_j}{\partial \xi} \right), \quad i = 0, 1, \ldots, N.$$

Using (33), (35) and $\partial q_i/\partial \xi \equiv \partial t_i/\partial \xi + \partial p_i/\partial \xi$, it can be seen that the first term on the right hand side of (31) is equal to $\sum_{i=0}^{N}(\partial q_i/\partial \xi) \cdot (\sum_{h=1}^{H} \partial \bar{x}_i^h/\partial \xi)$, while the second term on the right hand side of (31) is equal to $-\sum_{i=0}^{N}(\partial p_i/\partial \xi)(\partial y_i/\partial \xi)$. If we further assume that the direction of tax change is chosen so that[10]

$$(36) \qquad \partial u^h/\partial \xi = 0, \quad h = 1, 2, \ldots, H,$$

then upon comparing (34) and (35), we find that $\partial \bar{x}_i^h/\partial \xi = \partial x_i^h/\partial \xi$. Under these conditions the Boiteux measure of welfare loss (32) can be rewritten as:

$$(37) \qquad -\frac{1}{2}\xi^{*2}\left\{ \sum_{i=0}^{N} \frac{\partial q_i}{\partial \xi} \left(\sum_{h=1}^{H} \frac{\partial x_i^h}{\partial \xi} \right) - \sum_{i=0}^{N} \frac{\partial p_i}{\partial \xi} \frac{\partial y_i}{\partial \xi} \right\}.$$

Since $\sum_{h=1}^{H} \partial x_i^h/\partial \xi = \partial y_i/\partial \xi$ for $i = 0, 1, \ldots, N$ and $\partial q_i/\partial \xi = \partial p_i/\partial \xi + \partial t_i/\partial \xi$, (37) simplifies to the *Hotelling measure of welfare loss*:

$$(38) \qquad L_H \equiv -\frac{1}{2}\xi^{*2} \sum_{i=0}^{N} \left(\frac{\partial t_i}{\partial \xi} \right) \left(\frac{\partial y_i}{\partial \xi} \right).$$

Thus the welfare loss due to a movement away from an initial system of taxes on fixed factors only is approximately equal to minus one half the sum over i of the monetary value of the ith commodity tax times the induced change in the demand for (or the supply of) commodity i, provided that the direction of tax change is such that the real income of every consumer is unchanged to the first order (i.e., (36) is satisfied). If the last proviso is not satisfied, then L_B defined by (31) and (32) does *not* simplify to (37) or (38).

The approximate loss measure defined by (38) is essentially due to Hotelling [**22**, p. 254] and has been rederived by Boiteux [**3**, p. 130], Harberger [**16**, p. 70, and **17**, p. 790] and others under various regularity assumptions.

4. THE COEFFICIENT OF RESOURCE UTILIZATION

Let $\bar{u}$, $\bar{p}$, $\bar{t}_0$, $\bar{t}.$, and $\bar{\tau}$ denote an initial equilibrium for our economy; i.e., these variables satisfy equations (15)–(18). Suppose that we scaled each consumer's vector of fixed factors v^h by the positive scalar ρ. We could now attempt to find

[10] Recalling (29), this means we will have to choose $\tilde{t}.$ and $\tilde{\tau}$ to be such that $\nabla_\xi u = 0_H$. In general, this requires that $N + M \geqq H$.

the smallest ρ such that the deflated aggregate fixed factor vector, $\rho v \equiv \rho \sum_{h=1}^{H} v^h$, is consistent with a competitive equilibrium with government expenditures; i.e., we could attempt to minimize ρ with respect to $\rho, p_{.}, t_0, t_{.}$, and τ subject to the following H budget constraints:

$$(39) \qquad m^h(\bar{u}^h, p_0 + t_0, p_{.} + t_{.}) = \rho v^{hT}(\nabla_v \pi(p_0, p_{.}, \rho v) - \tau)$$
$$= \rho v^{hT}(\nabla_v \pi(p_0, p_{.}, v) - \tau), \quad h = 1, \ldots, H,$$

where the second equality in (39) follows from the linear homogeneity of π in v, and subject to the following $N + 1$ demand equals supply constraints:

$$(40) \qquad \sum_{h=1}^{H} \nabla_{q_i} m^h(\bar{u}^h, p + t_0, p_{.} + t_{.}) + x_1^0 = \nabla_{p_i} \pi(p_0, p_{.}, \rho v)$$
$$= \rho \nabla_{p_i} \pi(p_0, p, v), \quad i = 0, 1, \ldots, N.$$

The solution $\rho^*(\bar{u})$ to the above constrained minimization problem is essentially[11] Debreu's [**4**, **5**] *coefficient of resource utilization*. Debreu shows that $\rho^*(\bar{u}) = 1$ if $\bar{u}$ corresponds to an initial equilibrium which is Pareto optimal. On the other hand, if $\bar{u}$ corresponds to a tax ridden competitive equilibrium, then $\rho^*(\bar{u}) \leqq 1$ and Debreu's [**5**, p. 16] *loss of efficiency* measure associated with the situation is $1 - \rho^*(\bar{u})$. The interpretation of this measure is that the government could confiscate and throw away the fraction $1 - \rho^*(\bar{u})$ of the economy's initial endowment of fixed factors and still achieve the same level of utility for each consumer (by a suitable change in tax policy) as in the initial tax ridden equilibrium. Thus $1 - \rho^*(\bar{u})$ is a measure of the deadweight loss due to distortionary taxation similar in nature to the Hotelling-Boiteux measure studied in the previous section.[12] However, Debreu [**4**, p. 286] notes that these latter measures of loss, which can be related to Hicksian consumer surplus concepts, suffer from the following defect: given two initial equilibria for the economy which have the *same* utility levels (i.e., let $\bar{u}, \bar{p}'_{.}, \bar{t}'_0, \bar{t}'_{.}, \bar{\tau}'$ and $\bar{u}, \bar{p}''_{.}, \bar{t}''_0, \bar{t}''_{.}, \bar{\tau}''$ each satisfy (15)–(18)), then the Hicksian measures of loss for the two equilibria will *not* generally be equal. Debreu's measure of loss does not suffer from this defect.

It should also be noted that Debreu's measure of loss can be regarded as a variant of a loss measure due to Allais [**1**, **2**]: in fact if there is only a single

[11] Debreu employs a primal formulation rather than our dual formulation (which utilizes the expenditure functions m^h and the variable profit function π), but more importantly, Debreu does not require the restrictions (39); instead he assumes that the government can control the distribution of income directly. We assume that each consumer's endowment of fixed factors v^h is deflated by ρ.

[12] Debreu [**4**, p. 285] notes that his measure of loss actually covers losses due to: (i) underemployment of physical resources, (ii) inefficiency in production, and (iii) imperfection of economic organization. Since we use a dual approach, we cannot measure losses due to (i) and (ii), a disadvantage to the use of duality theory. However, as Debreu [**4**, p. 286] notes, the third kind of loss "is the most subtle (in fact, perhaps hardly conceivable to the layman) and therefore the one for which a numerical evaluation is the most necessary."

primary factor ($M = 1$), then the Debreu and Allais measure can be made to coincide.[13]

We can consider the direction of tax change problem in the context of the coefficient of resource utilization. Suppose that $u, p_{.}, t_0, t_{.}, \tau$ and $\rho = 1$ satisfy equations (39) and (40). Then $u, p_{.}, t_0, t_{.}$, and τ will satisfy equations (15)–(18) and thus will correspond to an initial tax ridden competitive equilibrium. We ask whether the government can change tax rates and simultaneously confiscate fixed factors in such a way so that each consumer stays at his initial utility level. Upon totally differentiating (39) and (40) with respect to $\rho, p_{.}, t_0, t_{.}$, and τ, and remembering that $\rho = 1$ initially, we obtain the following system of equations which is approximately valid for small changes in the $1 + N + M - H$ exogenous tax variables:

$$(41) \qquad A_0 \Delta\rho = B_0 \Delta p_{.} + B_1 \Delta t_0 + B_2 \Delta t_{.} + B_3 \Delta\tau$$

where the matrices $[B_0, B_1, B_2, B_3] \equiv B$ are defined below (19),

$$A_0 \equiv \left[-v^{1T}(r-\tau), \dots, -v^{HT}(r-\tau), -y_0, -y_{.}^T \right]^T,$$

and $y^T \equiv [y_0, y_{.}^T]$ is defined by (1). Note that the right-hand side of (41) is the same as the right-hand side of (19). In order to justify rigorously (41), we make the following assumption which is analogous to assumption (20) made in Section 2:

$$(42) \qquad \left[A_0, -B_0, -B^* \right]^{-1} \quad \text{exists}$$

where B^* is an $H + 1 + N$ by H matrix which consists of H of the columns of $[B_1, B_2, B_3]$. Thus we consider $\rho, p_{.}$, and the H tax rates which correspond to the columns included in B^* to be endogenous variables and the remaining $1 + N + M - H$ tax rates to be exogenous. Thus we require $N + M \geqq H$.

The counterpart to the strict Pareto improving tax change studied by Hahn **[15]**, Guesnerie **[14]**, Harris **[18]**, Dixit **[12]**, Weymark **[32, 33]**, and Diewert **[10]** is now an *endowment reducing tax change*,[14] and it should be obvious that (42) plus the following condition is *sufficient* for one to exist:

$$(43) \qquad \exists z^1, z^2 \text{ such that } (-1)z^1 > 0 \quad \text{and} \quad A_0 z^1 = B z^2.$$

Obviously, if an endowment reducing tax change exists, then the initial system of taxes is wasteful or inefficient in the sense that resources could be thrown away but yet no household's real income need decline.

Application of Motzkin's Theorem of the Alternative to (43) yields the following equivalent *sufficient condition for the existence of an endowment reducing tax*

[13] Both Debreu [4] and Boiteux [3] acknowledge their indebtedness to Allais, who in turn cites the pioneering contributions of Pareto.

[14] Formally, an endowment reducing tax change exists if there exist $\bar{\rho} < 1, \bar{p}_{.}, \bar{t}_0, \bar{t}_{.}$, and $\bar{\tau}$ such that $\bar{\rho}, u, \bar{p}_{.}, \bar{t}_0, \bar{t}_{.}$, and $\bar{\tau}$ satisfy (39) and (40) where $\rho = 1, u, p_{.}, t_0, t_{.}$, and τ are the initial equilibrium values which satisfy (39) and (40).

change:

(44) There is no solution w to $w^T B = 0^T_{N+1+N+M}$ and $w^T A_0 = -1$.

COROLLARY: *If the rank of B is $H + N + 1$, then* (44) *is satisfied.*

The negations of (43) and (44) yield the following necessary conditions for the nonexistence of an endowment reducing tax change which can be interpreted as *necessary conditions for the initial system of taxes to be Debreu optimal*:[15]

(45) There is no solution z^1, z^2 to $(-1)z^1 > 0$ and $A_0 z^1 = Bz^2$;

(46) $\exists w$ such that $w^T B = 0^T$ and $w^T A_0 = -1$.

Conditions (44) and (46) can be used in order to prove counterparts to Propositions 1 to 4 in Diewert [**10**].

PROPOSITION 1: *Suppose that $q_0 \pi_{0.} + q_.^T \pi_{..} = 0_N^T$ and that $q^T y \neq 0$. Then the necessary condition for Debreu optimality* (46) *is satisfied.*

COROLLARY 1.1: *Suppose $t_0 = 0$, $t_. = 0_N$ so that the required government revenue is being raised by taxing only the fixed factors. Then the necessary condition for Debreu optimality* (46) *is satisfied.*

COROLLARY 1.2: *Suppose that $t = t_0 p$ so that the variable commodity taxes are proportional* (*this means labor services are being subsidized if $t_0 > 0$*). *Then the necessary condition for Debreu optimality* (46) *is satisfied.*

COROLLARY 1.3: *Suppose that the technology is Leontief so that $\pi_{..} = 0_{N\times N}$* (*which implies by* (5) *that $\pi_{0.} = 0_N^T$ also*). *Then the necessary condition for Debreu optimality* (46) *is satisfied.*

PROPOSITION 2: *Suppose that $p_0 \sigma_{0.} + p_.^T \sigma_{..} = 0_N^T$. Then the necessary condition for Debreu optimality* (46) *is satisfied.*

COROLLARY 2.1: *If consumer preferences are Leontief so that $\sigma_{..} = 0_{N\times N}$* (*which implies using* (7) *that $\sigma_{0.} = 0_N^T$*), *then* (46) *is satisfied.*

PROPOSITION 3: *Suppose that*: (i) $[v^1, v^2, \ldots, v^H]$ *has rank H*, (ii) $\sigma_{..}^{-1}$ *exists, and* (iii) $q_0 \pi_{0.} + q_.^T \pi_{..} \neq 0_N^T$. *Then the sufficient condition for an endowment reducing tax change* (44) *is satisfied.*

[15] Formally, an initial equilibrium $\rho = 1$, u, $p_.$, t_0, $t_.$, τ is Debreu optimal if there does not exist $\bar{\rho} < 1$, $\bar{p}_.$, $\bar{t}_0$, $\bar{t}_.$, τ such that $\bar{\rho}$, u, $\bar{p}_.$, $\bar{t}_0$, $\bar{t}_.$, and $\bar{\tau}$ satisfy (39) and (40).

PROPOSITION 4: *Suppose that*: (i) $[v^1, v^2, \ldots, v^H]$ *has rank* H, (ii) π^{-1} *exists, and* (iii) $p_0\sigma_{0.} + p_.^T\sigma_{..} \neq 0_N^T$. *Then the sufficient condition for an endowment reducing tax change is satisfied.*

Further propositions can be proven when there are equality or inequality constraints on the tax instruments. However, we now attempt to find the first and second derivatives of ρ with respect to a direction of tax change.

As in Section 3, assume that the government can raise its required revenue by taxing fixed factors alone. Let $\rho^* = 1$, u^*, $p_.^*$, $t_0^* = 0$, $t_.^* = 0_N$, and τ^* denote the initial equilibrium values which satisfy equations (39) and (40). Assume (42) for this initial equilibrium. Let B^{**} denote the matrix which consists of the $1 + N + M$ columns of B which are not included in the matrix B^*. Let t_x denote the $1 + N + M - H$ dimensional vector of exogenous tax rates while t_e denotes the H dimensional vector of endogenous tax rates (the components of t_x and t_e together make up the components of the $1 + N + M$ tax rates in t_0, $t_.$, and τ). Now define the exogenous tax rates t_x to be linear functions of the scalar ξ in a manner analogous to equation (25):

$$t_x = t_x^* + \tilde{t}_x\xi \tag{47}$$

where t_x^* denotes the initial equilibrium values for the exogenous tax rates and the direction of tax change vector $\tilde{t}_x$ satisfies the restriction $\tilde{t}_x^T\tilde{t}_x = 1$. Using Assumption (42), ρ, $p_.$, and t_e can be defined as implicit functions of t_x and hence of the scalar variable ξ. The derivatives of our equilibrium variables with respect to ξ evaluated at $\xi = 0$ are:

$$\begin{aligned} &\partial p_0/\partial\xi \equiv 0, \quad \nabla_\xi u \equiv 0_H, \quad \nabla_\xi t_x = \tilde{t}_x \quad \text{and} \\ &\left[\frac{\partial\rho}{\partial\xi}, \nabla_\xi p_.^T, \nabla_\xi t_e^T\right]^T = \left[A_0, -B_0, -B^*\right]^{-1}B^{**}\tilde{t}_x. \end{aligned} \tag{48}$$

With the preliminaries disposed of, multiply the ith equation of (40) by p_i and sum the resulting equations. Making use of the linear homogeneity properties of m^h and π, we obtain the following equation:

$$\begin{aligned} &\sum_{h=1}^{H} m^h(u^h, p_0 + t_0, p_. + t_.) + \sum_{i=0}^{N} p_i x_i^0 - \rho\pi(p_0, p_., v) \\ &\qquad - \sum_{i=0}^{N} t_i \sum_{h=1}^{H} \nabla_{q_i} m^h(u^h, p_0 + t_0, p_. + t_.) = 0. \end{aligned} \tag{49}$$

Differentiate (49) with respect to ξ, regarding $\rho, p_.$, t_0, $t_.$, and τ as functions of ξ. After cancelling some terms and using equations (40), we get

$$\pi(p_0, p_., v)\frac{\partial\rho}{\partial\xi} = -\sum_{i=0}^{N} t_i \frac{\partial}{\partial\xi}\left[\sum_{h=1}^{H} \nabla_{q_i} m^h(u^h, + t_0, + t_.)\right]. \tag{50}$$

When $\xi = 0$, $t_0 = t_0^* \equiv 0$, and $t_. = t_.^* \equiv 0_N$ so that (50) reduces to $\pi(p_0^*, p_.^*, v)$ $\partial\rho/\partial\xi = 0$. Since we assume that $\pi(p_0^*, p_.^*, v) > 0$, we have

$$\text{(51)} \qquad \frac{\partial \rho}{\partial \xi} = 0$$

for all directions of tax change $\tilde{t}_x$ such that $\tilde{t}_x^T \tilde{t}_x = 1$. Now differentiate (50) again with respect to ξ and evaluate the resulting derivatives at $\xi = 0$. Using (51) and the identity we obtain when we differentiate the ith equation in (40) with respect to ξ, premultiplying the resulting equation by $\partial p_i/\partial\xi$ and summing over i, we find that

$$\text{(52)} \qquad \pi(p_0^*, p_.^*, v)\frac{\partial^2 \rho}{\partial \xi^2} = -\sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{\partial t_i}{\partial \xi} + \frac{\partial p_i}{\partial \xi}\right)\left(\sum_{h=1}^{H}\sigma_{ij}^h\right)\left(\frac{\partial t_j}{\partial \xi} + \frac{\partial p_j}{\partial \xi}\right) + \sum_{i=0}^{N}\sum_{j=0}^{N}\left(\frac{\partial p_i}{\partial \xi}\right)(\pi_{ij})\left(\frac{\partial p_j}{\partial \xi}\right)$$

where the derivatives $\partial t_i/\partial\xi$ and $\partial p_i/\partial\xi$ are defined in (48). Note that the right-hand side of (52) is nonnegative for all directions of tax change $\tilde{t}_x$. Thus the second order necessary conditions for ρ to be a minimum with respect to ξ are satisfied at the initial pure fixed factor tax equilibrium for all directions of tax change $\tilde{t}_x$.

When we increase ξ, we introduce distortionary commodity taxes and in order to keep every consumer at his initial utility level u^{h*}, we will generally have to increase the aggregate endowment of the economy; i.e., ρ will have to increase as ξ increases. An approximate measure of the monetary cost of the extra resources required in order to keep every real income constant when we move towards distortionary commodity taxation from an initially undistorted point by increasing ξ from 0 to ξ^* is

$$\text{(53)} \qquad L_D \equiv \frac{1}{2}\xi^{*2}\pi(p_0^*, p_.^*, v)\frac{\partial^2 \rho}{\partial \xi^2}$$

where $\pi(p_0^*, p_.^*, v)\partial^2\rho/\partial\xi^2 = r^T v \partial^2\rho/\partial\xi^2$ is defined by (52). The loss measure defined by (53), using (48) and (52), is essentially due to Debreu [**5**, p. 19]. Note that the Debreu measure of loss L_D has the same form as the Boiteux measure of loss L_B defined by (29), (31), and (32). However, the two measures do not coincide except under special circumstances: the problem is that the restrictions on the derivatives $\partial t_i/\partial\xi$ and $\partial p_i/\partial\xi$ given by (29) do not in general coincide with the restrictions given by (48). In view of (51), it can be seen that the derivatives $\partial t_i/\partial\xi$ and $\partial p_i/\partial\xi$ satisfy the restrictions

$$\text{(54)} \qquad 0_{H+1+N} = B_0 \nabla_\xi p_. + B_1 \nabla_\xi t_0 + B_2 \nabla_\xi t_. + B_3 \nabla_\xi \tau$$

when L_D is being calculated, but they only satisfy the restrictions

$$(55) \qquad A\nabla_\xi u = B_0\nabla_\xi p_. + B_1\nabla_\xi t_0 + B_2\nabla_\xi t_. + B_3\nabla_\xi \tau \quad \text{and} \quad 1_H^T\nabla_\xi u = 0$$

when L_B is being calculated. However, note that the Debreu and Hotelling measures of welfare loss coincide; i.e., $L_D = L_H$. Moreover if $H = 1$ so that there is only one consumer *or* if $\tilde{t}_.$ and $\tilde{\tau}$ in (29) are chosen so that $\nabla_\xi u = 0_H$, then $L_D = L_B$, and thus both the Boiteux and Debreu measures of welfare loss reduce to the Hotelling measure L_H defined by (38). Thus there is a close connection between the welfare loss measures due to Hotelling, Boiteux, and Debreu.

5. CONCLUSION

It should be noted that the formula developed in this paper are valid only for the infinitesimal changes in the exogenous tax variables. The question of how good are our formula when infinitesimal changes in taxes are replaced by finite tax changes has been extensively discussed by Shoven and Whalley [**29**, **30**], Whalley [**34**], Shoven [**28**], and Green and Sheshinski [**13**] in the context of several specific models. An advantage of our "local" approach as opposed to the Shoven-Whalley "global" approach is that we require only local information about preferences and technology. Of course, the disadvantage of the "local" approach is that the measures of welfare loss or gain may be quite inaccurate for large changes in tax rates.

Some additional limitations of our models are: (i) there is no foreign sector, (ii) government expenditures are held fixed, (iii) there are no distortionary taxes (such as manufacturer's excise taxes) within the production sector, (iv) there is no monopolistic or monopsonistic behavior, and (v) the model is atemporal—there is no savings or investment behavior in the economy. Our model can be extended to cover the omissions noted in (i) to (iv) above. For example, in order to deal with (iii), we need only decompose the private production sector into two sectors, one of which produces the taxed good and the other which uses the taxed good as an input. Markup monopolistic pricing behavior can readily be modelled by regarding the markups as taxes, where the tax revenue accrues to the monopolists (cf. Harberger [**16**, pp. 72–3] for an exposition of this approach). However, it appears to be extremely difficult to patch up (v).[16] The fundamental problem is that it is difficult to define optimality in an economy with incomplete markets, a situation which characterizes all real life economies.

An interesting issue that we have not explored is: under what conditions will the various loss measures be large? Inspection of the Boiteux measure of welfare loss L_B defined by (32) and (31) suggests that the loss will increase as substitutability both in consumption and production increases, i.e., as the aggregate

[16] See Dixit [**11**] on this topic. It should also be mentioned that difficulties (i), (ii), (iii), and (v) are all relaxed to varying degrees in the empirical work of Shoven and Whalley. Also Whalley [**35**] actually utilizes Debreu's coefficient of resource utilization in order to obtain numerical measures of the resources which could be saved by changing the tax system for some European countries.

Slutsky matrix σ becomes more negative semidefinite and as the net supply response matrix $[\pi_{ij}]$ becomes more positive semidefinite. However, this conjecture cannot be immediately substantiated in general since the price and tax derivatives, $\partial p_i/\partial\xi$ and $\partial t_i/\partial\xi$, which occur in (31) are complicated functions of σ and $[\pi_{ij}]$ (recall (29)). In the case of one consumer ($H = 1$), two variable goods ($N = 1$), and one fixed factor ($M = 1$), the above conjecture is true: the loss turns out to be a linearly homogeneous function in $\pi_{11} \geqq 0$ and $\sigma_{01}^1 \geqq 0$ which equals zero if either $\pi_{11} = 0$ or $\sigma_{01}^1 = 0$. Thus the loss can be very large if both π_{11} and σ_{01}^1 are large and positive. Note also that $L_B = L_H = L_D$ in the one consumer case so that all three loss measures will be large if there is a great deal of substitutability both in production and consumption (at least if $N = 1$ and $M = 1$).

To conclude, we have shown that the welfare loss measures suggested by Hotelling, Boiteux, and Debreu, L_H, L_B, and L_D respectively, are all closely related. In fact in our model, $L_H = L_D$ and under certain conditions, $L_H = L_D = L_B$. This seems somewhat remarkable since the Boiteux and Debreu measures were developed originally from a totally different perspective.

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Manuscript received August, 1978; final revision received February, 1980.

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