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SMOOTH PREFERENCES[1]

BY GERARD DEBREU[2]

IN AN ATTEMPT to explain the observed state of an economy as an equilibrium resulting from the interaction of agents with partially conflicting aims, a mathematical model $\mathscr{E}$ of that economy is constructed. We will denote the set of equilibria of $\mathscr{E}$ by $E(\mathscr{E})$. Three basic questions must then be asked about the adequacy of the model $\mathscr{E}$. Before formulating them, however, I ought to stress their generality. Although in the following discussion, reference will be made only to the concept of Walrasian equilibrium of an exchange economy with a finite set of private commodities, these questions apply to any equilibrium concept introduced in the mathematical study of social systems.

1. NON-EMPTINESS OF $E(\mathscr{E})$

Since the purpose of the model $\mathscr{E}$ is to explain economic equilibrium, it is adequate only if $E(\mathscr{E})$ is not empty. This is the existence problem to which A. Wald [**39**] addressed himself in 1935–1936, and to which numerous contributions were made during the last twenty years. The main tools for its solution were provided by algebraic topology in the form of fixed point theorems. Rather than lingering on this first question, which is accessory to my main object, I refer to the recent extensive account by K. J. Arrow and F. H. Hahn [**2**], to which may be added the articles by R. J. Aumann [**4**], D. Schmeidler [**35**], and W. Hildenbrand [**23**] about economies with a measure space of agents, and the monograph by H. Scarf and T. Hansen [**34**] about the computation of economic equilibria.

2. DISCRETENESS OF $E(\mathscr{E})$

Ideally the set $E(\mathscr{E})$ would have exactly one element. A significant amount of work has also been done on the problem of uniqueness of equilibrium (K. J. Arrow and F. H. Hahn [**2**, Ch. 9] give a comprehensive survey of this question), but no very satisfactory solution has been offered up to now. The assumptions made on $\mathscr{E}$ in order to obtain a unique equilibrium are exceedingly strong. Indeed it suffices to look into Edgeworth's box to find economies with two commodities and two consumers having no pathologies, exhibiting several equilibria, even a continuum of equilibria.

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An alternative approach is to introduce an appropriate topology on the set of states of $\mathscr{E}$ and to make the weaker requirement of *local* uniqueness. A state of $\mathscr{E}$ having a neighborhood in which it is the unique element of $E(\mathscr{E})$ provides a determinate explanation of equilibrium insofar as only small changes in that state can take place. Conversely if an equilibrium e is not locally unique, there are arbitrarily small perturbations of the state of the economy from e giving rise to no tendency for the economy to return to e. Thus one is led to search for assumptions on $\mathscr{E}$ insuring that all the elements of $E(\mathscr{E})$ are locally unique, i.e., that the set $E(\mathscr{E})$ is discrete. Actually this second adequacy requirement will usually appear in a stronger form, for in almost every case (possibly in every case) in which an equilibrium existence theorem has been proved for the economy $\mathscr{E}$, the set of equilibrium price vectors of $\mathscr{E}$ was compact. But for a compact set, discreteness is equivalent to finiteness.

Instead of requiring that all the elements of $E(\mathscr{E})$ be locally unique, one might be satisfied with a model $\mathscr{E}$ for which there is at least one locally unique equilibrium, but the conditions one is led to impose on $\mathscr{E}$ in order to obtain the latter imply the former. These conditions are differentiability assumptions (for instance on individual demand functions), and here the main mathematical tools for the solution of the problem of discreteness of the set of equilibria have been provided by differential topology. The results obtained in the last two years, G. Debreu [**11**], F. Delbaen [**14**], E. Dierker [**15**], E. and H. Dierker [**16**], K. Hildenbrand [**21**, **24**, Sec. 2.5], T. Rader [**30**, **31**], and S. Smale [**37**] are of the following nature. A precise definition of a well-behaved or *regular* economy is given such that every regular economy has a discrete set of equilibria; *critical* means non-regular. An appropriate topology is introduced on the set $\mathscr{S}$ of economies and the set $\mathscr{C}$ of critical economies is shown to be "negligible." If $\mathscr{S}$ is finite-dimensional (e.g., the set of economies with a finite set of agents, fixed preferences, and variable endowments), "negligible" means closed and of Lebesgue measure zero. When $\mathscr{S}$ is infinite-dimensional (which is the case in the above example as soon as preferences vary in an infinite-dimensional space, or the set of agents is infinite), in the situations studied so far the critical set $\mathscr{C}$ was shown to be "negligible" in the sense that it is closed and has an empty interior or, in the situation investigated by S. Smale [**37**], in the sense that it is contained in a countable union of closed sets with empty interiors.

The comments made earlier on the problem of global uniqueness of equilibrium call for a digression. Consider economies with l commodities. The characteristics of an agent a are his demand function f_a and his endowment ω_a. Denoting by P the set of strictly positive vectors in R^l, by S the set of vectors in P whose lth coordinate is unity, and by L the set of strictly positive reals, we assume that f_a establishes a one-to-one correspondence between $S \times L$ and P, that $p \cdot f_a(p, w) = w$ for every pair of a price vector p in S, and of a wealth w in L, and that f_a is continuously differentiable. The set of demand functions satisfying these conditions is denoted by $\mathscr{D}$. We assume that ω_a belongs to P. Thus a is an element of $A = \mathscr{D} \times P$ which we endow with a suitable topology. An economy $\mathscr{E}$ is identified with a positive measure ν on A such that $\nu(A) = 1$. The set of these economies (measures) is also endowed with a suitable topology. Now if all the agents of the economy are identical, i.e., if the measure ν is concentrated on one point $(\bar{f}, \bar{\omega})$ of A, a trivial case of global uniqueness of equilibrium obtains. Every agent consumes the commodity vector $\bar{\omega}$, and the unique equilibrium price vector is $\bar{p}$ such that $\bar{f}(\bar{p}, \bar{w}) = \bar{\omega}$. Assume in addition that the Jacobian determinant of $\bar{f}$ is different from zero at $(\bar{p}, \bar{w})$, a condition we will discuss in detail below. Finally let $p \mapsto F(p)$ be the function from S to R^{l-1} defined by $F_i(p) = \bar{f}_i(p, p \cdot \bar{\omega})$ for $i = 1, \ldots, l-1$. It is not difficult to prove that the Jacobian determinant of F at $\bar{p}$ equals the Jacobian determinant of $\bar{f}$ at $(\bar{p}, \bar{w})$. Therefore the economy with identical agents described above satisfies the definition of regularity of F. Delbaen [**14**, 4.12] and K. Hildenbrand [**24**, Sec. 2.5]. Consequently it has a neighborhood of economies also possessing a unique equilibrium. In other words, economies with sufficiently similar agents, in the precise sense we have given, have exactly one equilibrium. One may hope to go beyond this general assertion about global uniqueness by studying specific classes of economies (measures on A) exhibiting concentration around a central point. Indeed the possibility of deriving strong conclusions from the investigation of specific classes of measures on A seems to be a, so far largely unfulfilled, promise of the measure-theoretical approach in equilibrium analysis.

3. CONTINUITY OF E

The data of the model $\mathscr{E}$ (endowments, preference relations, or demand functions) cannot be exactly observed. If the correspondence E is not continuous at $\mathscr{E}$, an arbitrarily small error of observation on $\mathscr{E}$ yields an essentially different set of predicted equilibria. In such a situation the explanatory value of the model $\mathscr{E}$ seems to be limited. Therefore, the third adequacy requirement is that the correspondence E be continuous. The preceding considerations, which are common in the study of physical systems, apply with even greater force to social systems.

Actually this third requirement is closely related to the second, for in the papers listed in Section 2 that investigate this problem, it is shown that E is continuous on the set of regular economies. It is also related to studies of limit theorems and of hemi-continuity properties for various economic equilibrium concepts. References for the two concepts of the core and of Walrasian equilibrium are F. Y. Edgeworth [**18**], M. Shubik [**36**], H. Scarf [**33**], G. Debreu and H. Scarf [**12, 13**], Y. Kannai [**26**], W. Hildenbrand [**22, 24**], W. Hildenbrand and J. F. Mertens [**25**], K. J. Arrow and F. H. Hahn [**2**], B. Grodal [**19**], T. Bewley [**6**], and D. J. Brown and A. Robinson [**7, 8**].[3]

With the motivation of the last two adequacy requirements, I now propose to study from the differentiable viewpoint the three main concepts of the theory of consumer behavior: (i) preference relations, (ii) demand functions, and (iii) utility functions. The third is unsatisfactory as a primitive concept, for the axioms of the theory must be formulated in terms of observable choices made by a consumer among commodity vectors, and the existence of a utility function with specified

[3] A more complete bibliography will be found in W. Hildenbrand [**24**].

properties (e.g., a twice continuously differentiable utility function) must be derived from these behavior axioms. Revealed preference theory chooses the second as a primitive concept. We will choose the first.

The study of differentiable preference relations, which goes back to G. B. Antonelli's article [**1**] of 1886, has given rise to an extensive literature, completely surveyed by L. Hurwicz [**9**, Ch. 9]. Before presenting the approach to differentiable preferences in Antonelli's tradition, let us note that *henceforth consumption vectors will belong to P, the interior of the positive cone of R^l*. A description of the preferences of a consumer can be given by specifying for every x in P, a non-zero vector $g(x)$ in R^l, the intended interpretation for which is that the hyperplane $H(x)$ through x orthogonal to $g(x)$ is tangent at x to the indifference hypersurface of x, and $g(x)$ indicates a direction of preference. We normalize $g(x)$ by requiring that $\|g(x)\| = 1$, where the vertical bars denote Euclidean norm, and we assume that the function g from P to the unit sphere of R^l is of class C^1 (continuously differentiable). A basic problem is then the existence of a utility function u from P to R, of class C^2 (twice continuously differentiable), and such that its derivative Du is everywhere a strictly positive multiple of g, i.e., such that everywhere in P,

$$(1) \qquad Du = \lambda g,$$

where λ is a function from P to the set of strictly positive real numbers. By equating the partial derivatives $\partial_i \partial_j u$ and $\partial_j \partial_i u$, by writing similar equalities for the pairs (j, k) and (k, i) of indices, and by eliminating λ and its first partial derivatives, one obtains the following necessary conditions on g for the existence of functions u and λ respectively of class C^2 and C^1 satisfying (1) on P:

$$(2) \qquad \forall (i, j, k),\ g_i(\partial_j g_k - \partial_k g_j) + g_j(\partial_k g_i - \partial_i g_k) + g_k(\partial_i g_j - \partial_j g_i) = 0.$$

However the question of the sufficiency of these conditions (which are clearly not independent) is a less trivial matter. It is possible to prove that (2) implies for every point x^0 of P, the existence of an open neighborhood V of x^0, and of a utility function u of class C^2 defined and satisfying (1) on V. This result, whose long history in economic theory is told in detail in L. Hurwicz [**9**, Ch. 9], can be established as an application of a standard theorem of differential topology (N. J. Hicks [**20**, Sec. 9.1]), intimately related to the theorem of Frobenius on total differential equations (J. Dieudonné [**17**, Sec. 10.9]). The function H introduced earlier, associating with every x in P, the hyperplane $H(x)$ in R^l, is an $(l - 1)$-dimensional distribution of class C^1. Consider now two vector fields X and Y of class C^1 defined on an open subset U of P and lying in H, i.e., for every x in U, $X(x) \in H(x)$ and $Y(x) \in H(x)$, and check that the Lie bracket of X and Y, i.e., the vector $[X, Y]$ of R^l whose ith coordinate is $X \cdot DY_i - Y \cdot DX_i$, also lies in H. Thus the distribution H is involutive, and the first theorem of [**20**, Sec. 9.1] applies (the proof given in [**20**] is written for a C^∞ distribution but can be transposed for a C^1 distribution). Given a point x^0 of P, there is an open neighborhood V of x^0 and a C^2 coordinate system $y_1, \ldots, y_l$ defined on V such that the subsets of V on which y_l is constant are integral manifolds of H. The function y_l is a utility function on V.

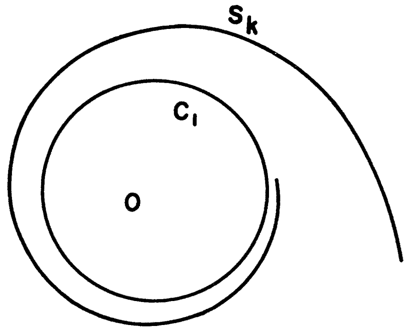

FIGURE 1.

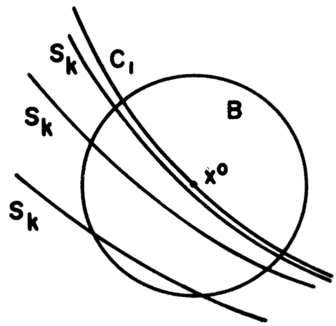

FIGURE 2.

But conditions (2) do *not* imply the existence of a utility function u of class C^2 defined and satisfying (1) everywhere on P. As P. A. Samuelson [**32**] used spirals around a point in his discussion of local integrability, we will use spirals around a circle to show that global integrability is not a consequence of local integrability. Consider in R^2 the circles C_r with center 0 and radius r such that $0 < r \leqslant 1$, and the spirals S_k with equations $\rho(\theta) = ke^{-\theta} + 1$ in polar coordinates, where the angle θ is measured in radians and $e^{-2\pi} < k \leqslant 1$. For every such k, the spiral S_k winds around, and approaches, C_1 as θ increases indefinitely. Through every point x of R^2 different from 0, there is a unique curve of the above family. Let $g(x)$ be its unit normal pointing toward 0. Clearly g satisfies (2). However there can be no C^2 real-valued function $\bar{u}$ satisfying (1) on $R^2 \diagdown \{0\}$. Assume that there is such a function $\bar{u}$, and examine a point x^0 of the circle C_1, a small open ball B with center x^0, and a spiral S_k. The function $\bar{u}$ has the same constant value on every arc of S_k in B, while (1) implies that the value of $\bar{u}$ increases as one moves toward 0 in B, a contradiction. Since this example does not satisfy the condition imposed earlier that consumption vectors belong to the open positive cone P of R^2, we translate Figure 1 from 0 to a point $0'$ with both coordinates strictly greater than 1, and restrict our attention to those parts of the spirals lying in P.

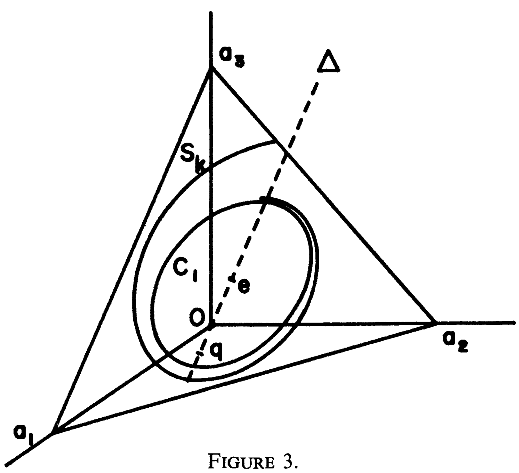

FIGURE 3.

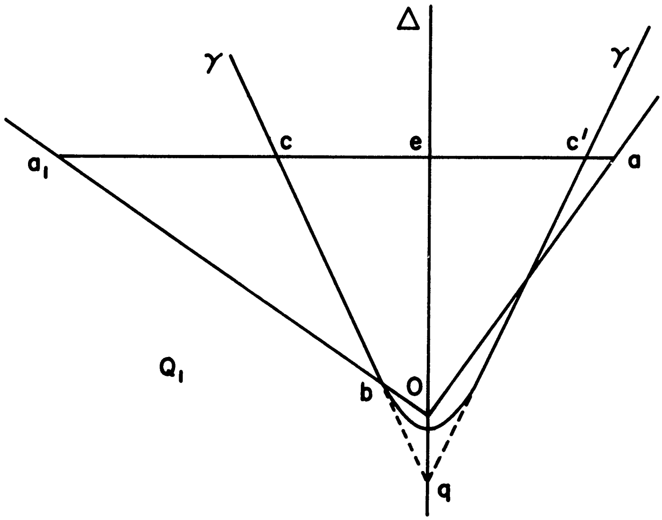

FIGURE 4.

The preceding example is still not fully satisfactory, for the vector field g cannot be extended continuously to 0′; in economic terms, 0′ is a satiation point. To overcome this difficulty, consider the equilateral triangle with vertices a_1, a_2, a_3 on the coordinate axes of R^3 at distance 3 from the origin, $T = \{x \in R^3 | x \gg 0, x_1 + x_2 + x_3 = 3\}$, where $x \gg 0$ means that all the coordinates of x are strictly positive; let Δ be the straight line in R^3 defined by the equations $x_1 = x_2 = x_3$; and let e be the point (1, 1, 1) where Δ intersects T. We now place Figure 1 in the plane of T with the center of the concentric circles at e, choose a point q of Δ with strictly negative coordinates, and construct for every spiral S_k, the cone Σ_k with vertex q generated by S_k. Figure 4 is drawn in the plane Q_1 containing Δ and the first coordinate axis. The segment a_1, a, with endpoints excluded, is the intersection of T with the plane Q_1. The points c, c' of the segment a_1, a at distance 1 from e are the intersection of the circle C_1 in T with the plane Q_1. The point b is the intersection of the two straight lines q, c and 0, a_1. Next we draw a smooth convex curve γ symmetric around Δ, coinciding with the straight line q, c above b, and intersecting Δ below 0. Denote by Γ_0 the revolution surface generated by rotating γ around Δ, and by Γ_k the surface obtained from Γ_0 by an upward translation parallel to Δ by a distance $k \geqslant 0$. A point x of P, the open positive cone of R^3, belongs to exactly one surface of either the family $\{\Sigma_k\}$, or the family $\{\Gamma_k\}$. The unit normal $g(x)$ to that surface at the point x, pointing toward Δ satisfies conditions (2) everywhere on P. As before, by considering a point x^0 of C_1, we show that there cannot be a C^2 real-valued function $\bar{u}$ satisfying (1) on P.

It is possible however to prove that if for every x in P, the open positive cone of R^l, one has $g(x) > 0$, then local integrability implies global integrability. The inequality $g(x) > 0$ means that the vector $g(x)$ has all its coordinates non-negative, and is different from 0; in economic terms, preferences are locally monotone. As a consequence of (2), there is through every point of P, a maximal connected integral

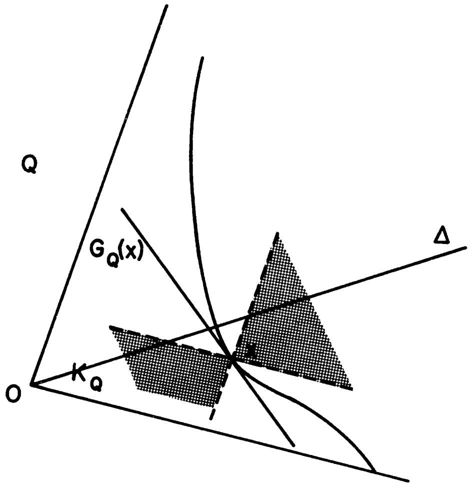

FIGURE 5.

manifold of the distribution H (L. Auslander and R. E. MacKenzie [**5**, Sec. 8.8] or S. Sternberg [**38**, Sec. 3.5]). To study the structure of these manifolds, we examine their intersections with two-dimensional planes containing Δ, the straight line in R^l with equations $x_1 = x_2 = \ldots = x_l$. Let Q be such a plane, and let K_Q be the intersection of Q and P. Given a point x of K_Q, the hyperplane $H(x)$ supports the translate of P from 0 to x, since $g(x) > 0$. Therefore the straight line $G_Q(x)$, intersection of $H(x)$ and Q, supports the translate of K_Q from 0 to x. Now the intersection of Q and an integral manifold of the distribution H is an integral curve of the distribution G_Q. Clearly a connected integral curve of G_Q through x cannot intersect the translates of K_Q and $-K_Q$ from 0 to x (represented by the shaded angles of Figure 5); if it did, there would be a point on this curve at which the tangent would not support the translate of K_Q from 0 to that point. Consequently the maximal connected integral (MCI) curve of G_Q through x intersects any ray from 0 in K_Q in exactly one point; and either it tends to a straight line parallel to an edge of K_Q,

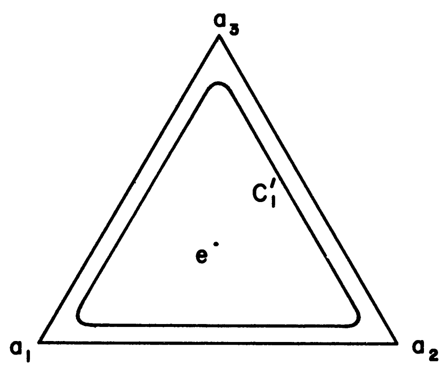

FIGURE 6.

or its closure intersects the boundary of K_Q. Given two MCI curves of G_Q, one is above the other in the sense that all the rays from 0 in K_Q intersect these two curves in the same order. Consider now a point y of Δ. As Q ranges over the set of planes containing Δ, the MCI curves of G_Q through y generate the MCI manifold of H through y. Given x in P, the MCI manifold of H through x intersects Δ in a unique point $U(x)$ with equal coordinates $u(x)$. The function u is of class C^2 and satisfies (1) everywhere on P (in particular, $Du(x) > 0$ for every x in P).

The preceding reasoning fails in the example illustrated by Figures 3 and 4 because for any plane Q containing Δ, there are MCI curves of G_Q which do not intersect Δ. A slight modification of that example also shows that local monotony of preferences is somehow required for local integrability to imply global integrability. To see this, it suffices to replace the circle C_1 by a smooth curve C_1' close to the boundary of the equilateral triangle T, to insert a suitable family of spirals between C_1' and the boundary of T, and a suitable family of closed curves inside C_1', and to choose q close to 0. In this manner one can approximate local monotony as closely as one wishes, and still not obtain global integrability.

Thus, following Antonelli's approach, we have been led to preferences characterized by *a function* $g > 0$ *of class* C^1 *from* P, *the open positive cone of* R^l, *to the unit sphere of* R^l, *satisfying* (2) *on* P. Let $\mathscr{G}$ be the set of functions g fulfilling all these conditions. A natural topology can be defined on this set $\mathscr{G}$ of preference relations, for instance as the topology of uniform convergence (or of uniform convergence on compact subsets of P) of g and of its l partial derivatives $\partial_i g$.

An alternative approach to smooth preference relations consists of making differentiability assumptions on their graphs. Given two open sets X and Y in R^n, a function h from X onto Y is a C^k-*diffeomorphism* if h is one-to-one, and both h and h^{-1} are of class C^k. A set M of points in R^n is a C^k-*hypersurface* if for every $z \in M$, there is an open neighborhood U of z, a C^k-diffeomorphism h of U onto an open set V in R^n, and a hyperplane H in R^n such that $M \cap U$ is carried by h into $H \cap V$. In other words, M is locally a hyperplane, up to a C^k-diffeomorphism. Let now $\precsim$ be a complete preference preordering on P, i.e., a complete, transitive, and reflexive binary relation on the open positive cone of R^l. Define the relation of strict preference $[x \prec y]$ by $[x \precsim y$ and not $y \precsim x]$, and the relation of indifference $[x \sim y]$ by $[x \precsim y$ and $y \precsim x]$. Let us also say that the preference relation is *continuous* if the set $\mathscr{P} = \{(x, y) \in P \times P | x \precsim y\}$ is closed in $P \times P$, and that it is *monotone* if $x \ll y$ implies $x \prec y$. *Henceforth we will assume that* $\precsim$ *is a monotone, continuous, complete preordering on* P. In these conditions, the boundary of $\mathscr{P}$ in $P \times P$ is the set $\mathscr{I} = \{(x, y) \in P \times P | x \sim y\}$. We will say that $\precsim$ *is a preference relation of class* C^k *if* $\mathscr{I}$ *is a* C^k-*hypersurface in* R^{2l}.

At this point, three ways of approaching the question of smooth preferences have been, explicitly or implicitly, mentioned. One can postulate (i) a monotone, continuous, complete preference preordering $\precsim$ on P of class C^2, (ii) a function $g > 0$ from P to the unit sphere of R^l of class C^1 satisfying (2) on P, or (iii) a real-valued utility function u on P of class C^2, satisfying $Du(x) > 0$ for every x in P. We will prove that these three postulates are equivalent.

It has been established above that (ii) implies (iii).

Next observe that (iii) implies (i). The only property of the preference preordering $\precsim$ which is not obvious is that it is of class C^2. Letting $v(x, y) = u(x) - u(y)$, we obtain $\mathscr{I} = \{(x, y) \in P \times P | u(x) - u(y) = 0\} = \{z \in P \times P | v(z) = 0\}$. Since Du is different from 0 on P, so is Dv on $P \times P$. Consequently at every point z of $\mathscr{I}$, there is a local C^2-diffeomorphism h of the desired type.

Finally to see that (i) implies (ii), consider a point x^0 in P, and let $z^0 = (x^0, x^0)$, which is a point of $\mathscr{I}$ by reflexivity of $\precsim$. As a consequence of (i), there is an open neighborhood V of z^0 in $P \times P$, and a C^2 real-valued function v on V such that $v(z) = 0$ is equivalent to $z \in \mathscr{I} \cap V$, $v(z) \leqslant 0$ is equivalent to $z \in \mathscr{P} \cap V$, and $Dv \neq 0$ in V. Symmetry of the relation $\sim$ and monotony of the relation $\precsim$ imply that for every (x, x) in V, $Dv(x, x)$ has the form $(q(x), -q(x))$ where $q(x) > 0$. Given any x^1 in P, the indifference class of x^1 is $\{x \in P | (x, x^1) \in \mathscr{I}\}$. It is obtained by taking the intersection of $\mathscr{I}$ and the linear manifold $\{(x, y) \in R^l \times R^l | y = x^1\}$, and projecting into the first factor space R^l. Since for every x in an open neighborhood U of x^0 such that $U \times U \subset V$, the normal $Dv(x, x)$ to $\mathscr{I}$ at (x, x) projects into the first factor space R^l as $q(x) > 0$, one obtains the unit normal $g(x)$ at x to the indifference hypersurface through x by dividing $q(x)$ by its Euclidean norm. Clearly g has all the properties listed in (ii).

One of the three preceding equivalent postulates, namely (i), will be made from now on. We will also assume that $\precsim$ is strongly convex, i.e., $[x \sim y,\ x \neq y$, and $0 < t < 1]$ implies $[x \prec tx + (1 - t)y]$, and that the closures of the indifference hypersurfaces of $\precsim$ do not intersect the boundary of P. Thus, in addition to the conditions already imposed, *henceforth we assume that* $\precsim$ *is a strongly convex preference relation of class* C^2, *and that the closures of its indifference hypersurfaces are contained in* P. It follows that given a price vector p in R^l such that $p \gg 0$ and $\|p\| = 1$, and a wealth w in R such that $w > 0$, there is a unique commodity vector $f(p, w)$ satisfying optimally the preferences $\precsim$ in the set $\{x \in P | p \cdot x \leqslant w\}$.

Studies of the discreteness of $E(\mathscr{E})$, the set of equilibria of the economy $\mathscr{E}$, and of the continuity of the correspondence E mentioned earlier emphasize the desirability of obtaining by the procedure of the last paragraph a continuously differentiable demand function f. The derivation of a C^1 demand function from a C^2 preference relation is also part of the general program of investigating the equivalence of the two approaches to the theory of consumer behavior from preference relations or from demand functions. Digressing on the importance of this program, let us note that the introduction of an infinite set A of agents to analyze the connection between the core of an economy and the set of its equilibria has led to a search for richer and richer mathematical structures on A. Thus in R. J. Aumann [**3**], A is a measurable space. In Y. Kannai [**26**], G. Debreu [**10**], and W. Hildenbrand [**22**], A becomes a metric measurable space. The natural next step is to endow A with an algebraic structure. Such a structure should help in the study of the specific classes of measures on A to which an allusion was made at the end of Section 2, as well as in the study of the smoothing effect of the aggregation of individual demands to which an allusion will be made later. Now although a satisfactory metric could be defined on A by taking preference relations as a primitive concept, there seems to be more hope of finding an economically meaningful algebraic structure on A by taking demand functions as primitive.

The assumptions made so far on the preference relation $\precsim$ do not imply the differentiability of the demand function f associated with $\precsim$, as an example of D. W. Katzner [**27**] shows. Our purpose now is to give an additional, necessary and sufficient, condition for f to be of class C^1. The discussion will be centered on a standard concept of differential geometry, the Gaussian curvature of a hypersurface in R^l, which can be defined as follows (N. J. Hicks [**20**, Sec. 2.2]). Let M be a C^2 hypersurface in R^l, let x^0 be a point of M, and denote the unit normal to M at a point x of M in a neighborhood of x^0 by $g(x)$. To define the Gaussian curvature of M at x^0, consider a parametrized curve γ in M through x^0 with tangent vector t at x^0. Associate with each point x of γ, the point $g(x)$ in the unit sphere M' of R^l. We obtain in this manner a parametrized curve γ' in M' through $g(x^0)$ with tangent vector t' at $g(x^0)$. The vector t' depends only on the vector t (and not on the parametrized curve γ). It is parallel to the hyperplane $H(x^0)$ tangent to M at x^0. The function $t \mapsto t'$ (the Weingarten map) is linear. By definition, its determinant is the Gaussian curvature of M at x^0.

To calculate the curvature $c(x^0)$ at a point x^0 of P of the indifference hypersurface through x^0, notice first that monotony and strong convexity of the relation $\precsim$ imply strong monotony, i.e., $g(x) \gg 0$ on P (or $x < y \Rightarrow x \prec y$). Next choose the first $l-1$ coordinates in R^l of a point of M, resp. of M', as a coordinate system in a neighborhood of x^0, resp. of $g(x^0)$. Thus $c(x^0)$ is the Jacobian determinant of the transformation $(x_1, \ldots, x_{l-1}) \mapsto (g_1(x), \ldots, g_{l-1}(x))$ where $x_l = \xi(x_1, \ldots, x_{l-1})$ is determined implicitly by $u(x_1, \ldots, x_l) = u(x^0)$. Therefore $c = |\partial_j g_i + \partial_l g_i \partial_j \xi|$ where $i, j = 1, \ldots, l-1$. Substituting $-g_j/g_l$ for $\partial_j \xi$, and performing straightforward determinant manipulations, we obtain

$$(3) \qquad c = -\begin{vmatrix} \partial_j g_i & g_i \\ g_j & 0 \end{vmatrix}, \qquad (i, j = 1, \ldots, l).$$

In this formula, the square $l \cdot l$ matrix $[\partial_j g_i]$ is bordered by a column of g_i, a row of g_j, and the number 0.

Given a unit vector p in P, denote by $\hat{p}$ the $(l-1)$-vector $(p_1, \ldots, p_{l-1})$. The demand function f corresponding to this particular representation of the price system is the inverse of the function $x \mapsto (\hat{g}(x), v(x))$ where x is a commodity vector in P and $v(x)$ is the inner product $g(x) \cdot x$. The function f^{-1} will be studied as the composition of the two functions $\alpha : x \mapsto (\hat{g}(x), u(x))$ and $\beta : (\hat{p}, y) \mapsto (\hat{p}, m(y))$ where p is a strictly positive unit vector in R^l, y is a real number in the range of u, and $m(y) = \min_{u(z) \geqslant y} p \cdot z$. The Jacobian determinant of α is

$$J(\alpha) = \begin{vmatrix} \partial_j g_i \\ \partial_j u \end{vmatrix},$$

where $i = 1, \ldots, l-1$, and $j = 1, \ldots, l$. By a simple manipulation of this determinant, we obtain $J(\alpha) = g_l \cdot c \cdot \|Du\|$. As for the Jacobian of β, it is $J(\beta) = Dm$. However the function m is the inverse of the function $w \mapsto \max_{p \cdot z \leqslant w} u(z)$ where w

is a strictly positive real number. By a simple and classical calculation, $Dm^{-1} = \|Du\|$. Therefore $J(\beta) = \|Du\|^{-1}$. Summing up, $J(f^{-1}) = J(\beta \circ \alpha) = g_l c$. Since α and β are of class C^1, if $J(\beta \circ \alpha) \neq 0$ at the point x^0, then the demand function f is of class C^1 at the point $f^{-1}(x^0)$. Conversely if f is of class C^1, then $\beta \circ \alpha$ is a C^1 diffeomorphism and, consequently, its Jacobian is nonsingular; see J. Milnor [**29**, p. 4]. In conclusion, the demand function f is of class C^1 at the point $f^{-1}(x^0)$ if and only if the Gaussian curvature at x^0 of the indifference hypersurface through x^0 is different from zero. Since c can be expressed in terms of the classical determinant

$$\delta = \begin{vmatrix} \partial_i \partial_j u & \partial_i u \\ \partial_j u & 0 \end{vmatrix}$$

according to the formula $c = -\delta/\|Du\|^{l+1}$, which is easily derived from (3), the preceding intuitive condition of non-zero curvature is directly related to the condition $\delta \neq 0$ given by D. W. Katzner [**27**] for the differentiability of demand. It is also equivalent to the "strong convexity hypothesis" of E. Malinvaud [**28**], to which I refer for a proof of this assertion.

At this point, we consider a preference relation $\precsim$ on P as well behaved if it is a monotone, convex, continuous, complete preordering of class C^2, if its indifference hypersurfaces have everywhere a non-zero curvature, and if their closures are contained in P. By definition, a preference preordering on P is convex if for every x in P, the set of commodity vectors at least as desired as x is convex. The conditions listed above clearly imply that $\precsim$ is strongly convex. To appraise the strength of these conditions, let us endow the set of continuous, complete preference relations on P with one of the metrics proposed so far (Y. Kannai [**26**], G. Debreu [**10**], and W. Hildenbrand [**22**]). As an extension of unpublished results of B. Grodal on the approximation of a convex preference relation by a sequence of strongly convex preference relations, one may conjecture that in this framework, a monotone, convex, continuous, complete preference relation on P can be approximated on compact subsets of P by a sequence of preference relations satisfying all the conditions listed at the beginning of this paragraph.[4]

Going further in this direction, one might search for a topology on the set $\mathscr{K}$ of preference relations satisfying all the conditions listed at the beginning of the last paragraph, with the exception of non-zero curvature of indifference hypersurfaces, such that the subset $\mathscr{K}^0$ of preference relations satisfying all those conditions is open and dense in $\mathscr{K}$. Since $\mathscr{K}$ can be identified with a subset of the set $\mathscr{G}$ defined at the end of the discussion of global integrability, the topology sought should be closely related to the topologies suggested for $\mathscr{G}$. In light of the results of F. Delbaen [**14**] and E. and H. Dierker [**16**] on the open density of the set of regular economies with a finite set of consumers, variable endowments, and variable C^1 demand functions, such a topology on $\mathscr{K}$ might easily yield a theorem of S. Smale's [**37**] type in which the set of regular economies is open dense in the space of economies although the demand functions may not be continuously differentiable everywhere, and in fact may not even be defined everywhere.

[4] Added in proof: This conjecture was proved independently by F. Delbaen and by A. Mas-Colell (who has also answered the question raised in the next two sentences).

Still another way to approach the study of regular differentiable economies is to postulate an atomless measure ν on the space A of economic agents defined as the cartesian product $P \times \mathscr{F}$, where the first factor is interpreted as the set of endowments, and $\mathscr{F}$ is the set of monotone, convex, continuous, complete preference relations on P, endowed with one of the metrics of Y. Kannai [**26**], G. Debreu [**10**], and W. Hildenbrand [**22**]. One expects[5] that if the measure ν is suitably diffused over the space A, integration over A of the demand *correspondences* of the agents will yield a total demand *function*, possibly even a total demand function of class C^1.

University of California, Berkeley

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[5] Added in proof: As R. J. Aumann and W. Hildenbrand pointed out to me, the formulation of this problem I gave in Barcelona was not satisfactory. I hope that the present one, in all its vagueness, will contribute to stimulation of interest in an important and difficult question.

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