American Economic Association

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Source: *The American Economic Review*, Vol. 66, No. 2, Papers and Proceedings of the Eighty-eighth Annual Meeting of the American Economic Association (May, 1976), pp. 280-287
Published by: American Economic Association
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THE APPLICATION TO ECONOMICS OF DIFFERENTIAL TOPOLOGY AND GLOBAL ANALYSIS

Regular Differentiable Economies

By GERARD DEBREU*

The recent introduction of differential topology into economics was brought about by the study of several basic questions that arise in any mathematical theory of a social system centered on a concept of equilibrium. The purpose of this paper is to present a detailed discussion of two of those questions, and then to make a rapid survey of some related developments of the last five years.

Let e be a complete mathematical description of the economy to be studied (e.g., for an exchange economy, e might be a list of the demand functions and of the initial endowments of the consumers). Assumptions made a priori about e (e.g., assumptions of continuity on the demand functions) define the space $\mathcal{E}$ of economies to which the study is restricted. By a state of an economy we mean a list of specific values of all the relevant endogenous variables (e.g., prices and quantities of all the commodities consumed by the various consumers). We denote by S the set of conceivable states. Now a given equilibrium theory associates with each economy e in $\mathcal{E}$, the set $E(e)$ of equilibrium states of e, a subset of S.

As a first test of the adequacy of this mathematical model, it must be possible to prove that for every element e of a sufficiently broad class $\mathcal{E}$, the set $E(e)$ is not empty. This is the existence problem that has been extensively studied during the last decades. Mentioning only the early contributions of John von Neumann and Abraham Wald, I refer to the comprehensive survey of the literature by Kenneth Arrow and F. H. Hahn. The mathematical tools for the solution were provided by

* Department of Economics, University of California, Berkeley. I gratefully acknowledge the support of the National Science Foundation in the preparation of this article.

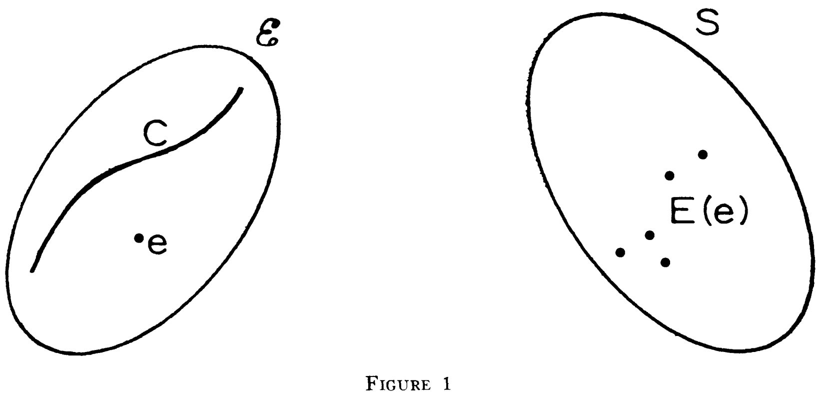

FIGURE 1

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algebraic topology in the form of fixed point theorems, or directly related results. An alternative approach of the last ten years, associated mainly with the name of Herbert Scarf, has consisted of developing efficient algorithms for the computation of an approximate equilibrium. The mathematical techniques used here were those of combinatorial topology.

Having obtained a general solution to the existence problem, one must investigate the structure of the set $E(e)$ of equilibria of the economy e. Consider an economy that has an equilibrium x such that in any neighborhood of x, there are infinitely many other equilibria. In this situation the explanation of equilibrium is essentially indeterminate, at least near x. Moreover the economic system e is unstable in the sense that arbitrarily small perturbations from x to a neighboring equilibrium induce no tendency for the state of the economy to return to x. It is therefore highly desirable to have an economy e for which the set of equilibria is *discrete*, i.e., such that for every equilibrium x in $E(e)$, there is a neighborhood of x in which x is the unique equilibrium of e. Unfortunately, even if every agent in the economy e is mathematically very well-behaved, one may obtain a set $E(e)$ that is not discrete (in the simplest case of an exchange economy with two consumers and two commodities, one can easily find in the associated Edgeworth box a set $E(e)$ made up of a continuum of points). The pathology is due to the manner in which the agents are matched, a situation entirely different from that of existence theory where it was possible to give general conditions on the behavior of each agent separately ensuring that the set $E(e)$ would not be empty.

The way out of this difficulty is provided by differential topology. It consists of making suitable differentiability assumptions on the functions entering the description of e (e.g., demand functions are assumed to be continuously differentiable), and to define a concept of ***regular*** economy such that (a) the critical set C of nonregular economies is a negligible subset of $\mathcal{E}$, and (b) every regular economy has a discrete set of equilibria.

Actually an adequate model e of the economy must have still another property. Specifically if e' is close to e, then one would like the set of equilibria $E(e')$ to be close to $E(e)$. Otherwise an arbitrarily small error in the determination of the characteristics of e would yield an entirely different set of equilibria, thus depriving the theory of much of its explanatory power. Therefore it is also desirable for the definition of regularity to be such that (c) in a neighborhood of a regular economy, the set of equilibria depends continuously on the economy. The questions of discreteness of the set of equilibria and of continuous dependence of the set of equilibria on the economy, or closely related questions, have a long history in the study of physical systems. The recent work of R. Thom has considerably extended the range of their applications, in particular to biological systems. These questions clearly have no less relevance for social systems.

The solution of the problem that has just been outlined rests on A. Sard's theorem which Thom once characterized as one of the three main results of Mathematical Analysis. Consider a continuously differentiable function f from the real line R to R, and define a *critical point* as a point x where the derivative of f vanishes. The set of critical points of f can obviously be large. In the extreme case of a constant function it is the whole of R. Define now a *critical value* of f as the image of a critical point. In Figure 2 three critical values y^1, y^2, y^3 are displayed.

One feels that the set of those critical values is necessarily small, and indeed Sard's theorem asserts that it is negligible. To be precise, it has (Lebesgue) ***measure***

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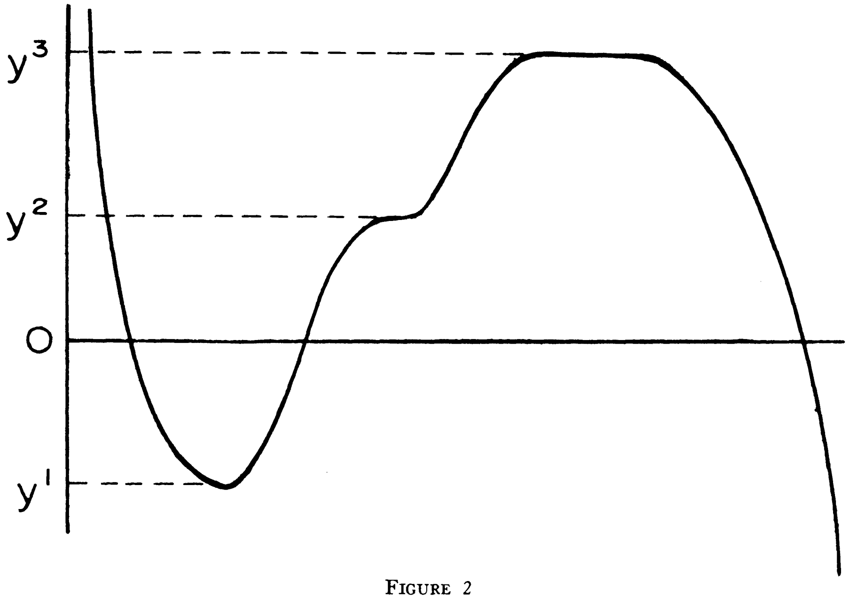

FIGURE 2

zero. In other words, given an arbitrarily small positive number ϵ, one can find a countable collection of intervals such that their union covers the critical set, and that the sum of the lengths of these intervals is smaller than ϵ. Sard's theorem holds as well for a continuously differentiable function f from an m-dimensional Euclidean space to an m-dimensional Euclidean space. Here a *critical point* of f is a point where the determinant of the Jacobian of f vanishes. As before, a *critical value* of f is the image of a critical point, and the set of critical values has measure zero, i.e., it can be covered by a countable collection of m-dimensional cubes of arbitrarily small total volume. A *regular value* of f is, by definition, a noncritical value.

Sard's theorem is valid in still more general conditions. In order to present a stronger version that we will use later on, we need the concept of a (differentiable) manifold. A function g from an n-dimensional Euclidean space to an n-dimensional Euclidean space is said to be a *differentiable isomorphism* if it is one-to-one, and if g, as well as its inverse g^{-1} are continuously differentiable. Differential topology studies properties that are invariant under differentiable isomorphisms, a principle

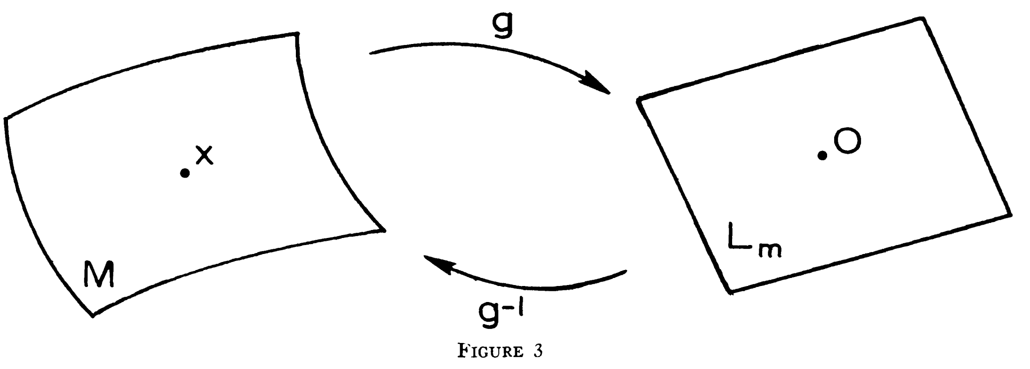

FIGURE 3

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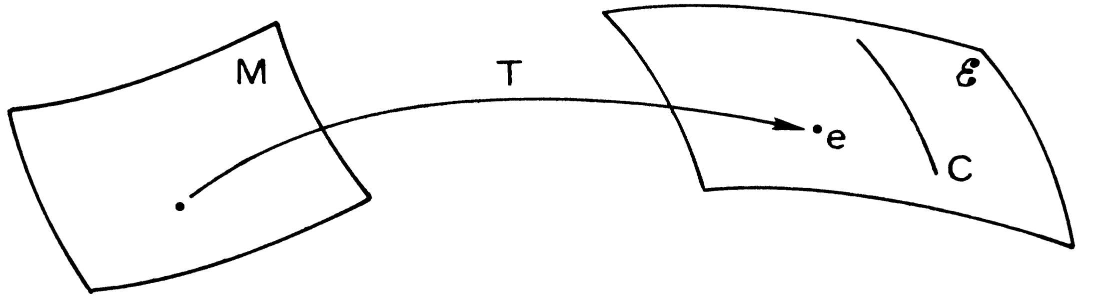

FIGURE 4

that leads to the following definition. A subset M of a Euclidean space L is an m-dimensional (differentiable) *manifold* if for each point x of M there is an m-dimensional linear subspace L_m of L such that a neighborhood in L of x and the part of M that it contains are differentiably isomorphic to a neighborhood in L of O and the part of L_m that it contains. In other words, at every one of its points, M is locally an m-dimensional Euclidean space up to a differentiable isomorphism. Clearly one can do differential calculus on manifolds exactly as on Euclidean spaces; one can also easily define a subset of measure zero of a manifold M. Then the statement of Sard's theorem becomes: if f is a continuously differentiable function from an m-dimensional manifold M_1 to an m-dimensional manifold M_2, the set of critical values of f has measure zero in M_2.

To illustrate these general principles, assume that the economy e can be characterized by finitely many real parameters, or more precisely, that $\mathcal{E}$ is a finite-dimensional manifold. A simple strategy for showing that almost every economy in $\mathcal{E}$ is well-behaved consists of introducing a manifold M of the same dimension as $\mathcal{E}$, and a continuously differentiable function T from M to $\mathcal{E}$ such that a *regular* economy, defined as a regular value of T, actually has properties (b) and (c). Sard's theorem implies that the set C of critical economies has measure zero.

To give an even more specific illustration, we consider an exchange economy with l commodities, m consumers, and fixed demand functions. Therefore the parameters of the economy are the initial holdings of each commodity by each consumer, a list e of lm positive numbers. We denote by $\mathcal{E}$ the set of those lm lists, and we observe for later use that the dimension of $\mathcal{E}$ is lm. The state of the economy is taken to be the price-system, i.e., a list p of l positive numbers. Since multiplying all prices by the same positive number does not affect the behavior of the agents, we can normalize the price-system and restrict it to belong to a manifold S, such that $\dim S = l-1$, for instance S may be the positive part of the unit sphere in the l-dimensional Euclidean space. Given the economy e and the price-system p, we write the l-list of the excesses of demand over supply on every market as $F(e, p)$, a vector in the l-dimensional space R^l. The price-system p is an equilibrium state if and only if

$$(1) \qquad F(e, p) = 0.$$

Given e, the set of p satisfying (1) is $E(e)$. However, the excess demand function F obeys Walras' law. Namely for every e and p, the value of the excess demand equals 0; i.e., $p \cdot F(e, p) = 0$. Consequently the equilibrium condition (1) is equivalent to equating to O the list $\hat{F}$ of the first $l-1$ components of F,

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(2) $\hat{F}(e, p) = 0.$

Now if the individual demand functions are continuously differentiable, we can folnow the strategy we have outlined above in several ways. An elementary treatment that does not use the concept of a manifold can be given as in Debreu (1970, 387–392). An alternative, and more satisfactory, solution takes as a central concept the set M of pairs (e, p) in the Cartesian product $\mathcal{E}\times S$ satisfying (2) (Stephen Smale 1974, pp. 1–14, Y. Balasko 1975, pp. 95–118). The space $\mathcal{E}\times S$ is of dimension $lm+l-1$, and the equilibrium condition (2) imposes $(l-1)$ restrictions on the pair (e, p). Therefore one may expect M to be a manifold of dimension lm, i.e., of the same dimension as $\mathcal{E}$. Indeed this is easily shown to be the case. The function T from the *equilibrium manifold* M to $\mathcal{E}$ could hardly be simpler; it is the transformation $(e, p)\mapsto e$, i.e., the projection from M into $\mathcal{E}$. A regular economy is then defined as a regular value of T. Equivalently e is a regular economy if for every (e, p) in M projecting into e, the projection of the tangent space of M at (e, p) covers $\mathcal{E}$. Three critical economies e^1, e^2, e^3 are displayed on Figure 5. Sard's theorem yields the conclusion that (a) the set C of critical economies has measure zero.

Given a regular economy e in $\mathcal{E}$ (as on Figure 5), one obtains the set $E(e)$ of equilibrium price-systems p by taking in M the set $T^{-1}(e)$ of inverse images of e by the projection T, and by projecting $T^{-1}(e)$ into S. At every point of $T^{-1}(e)$, the determinant of the Jacobian of T is different from O, and a direct application of the inverse function theorem yields that $T^{-1}(e)$ is a discrete subset of M. The pathologies associated with critical economies are made clear by Figure 5. For instance the economy e^1 has a discrete set of (two) equilibria, but a continuous displacement of the economy in a neighborhood of e^1 produces at e^1 a sudden change in the set of equilibria. In particular, consider e on the right of e^1 and the associated equilibrium p on the upper branch of M (as on Figure 5). When e moves left, follow p by continuity. As e crosses e^1, the equilibrium state of the system jumps to the lower branch of M. The economy e^3 has a continuum of equilibria and an isolated equilibrium. At the point e^3 one also observes a sudden change in the set of equilibria for a continuous displacement of the economy in a neighborhood of e^3. Although the economy e^2 is critical, it

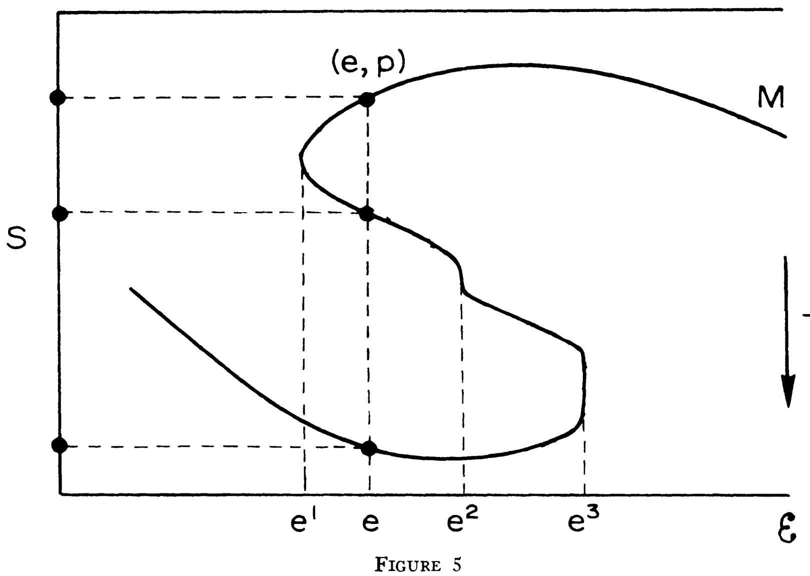

FIGURE 5

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has a discrete set of (three) equilibria, and in a neighborhood of e^3, the set of equilibria depends continuously on the economy.

One can obtain considerably stronger conclusions by making suitable assumptions on the behavior of the excess demand function F near the boundary of S. For instance, assume that given e, the excess demand $F(e, p)$ is unbounded when the price of a commodity tends to zero. In this case the set C of critical economies is *closed* in addition to being of measure zero, and is therefore negligible in a strong sense. Moreover (b') every regular economy has a *finite* set of equilibria, and (c') in a neighborhood of a regular economy, the set of equilibria depends in a *continuously differentiable* manner on the economy.

Property (c'), which strengthens property (c) of continuous dependence of the set of equilibria $E(e)$ on the economy e, has made it possible to answer the question of the rate of convergence of the core of an economy. Many authors from Edgeworth to W. Hildenbrand have shown with increasing generality that the core and the set of Walras equilibria of an economy tend toward each other as the number of agents increases in such a way that every one of them becomes insignificant. This precise formulation of the idea that an economy tends to become more competitive under these conditions raises the question of the rate of that convergence. A conjecture of Shapley (1975, pp. 345–51) has led to the proof (Debreu, 1975, pp. 1–7; B. Grodal, pp. 171–186) that outside the negligible critical set, the core and the set of Walras equilibria converge to each other at least as fast as the reciprocal of the number of agents converges to zero.

The results on negligible sets of critical economies that we have presented have been considerably extended by Balasko, Chichilnisky, Delbaen, E. and H. Dierker, Fuchs, K. Hildenbrand, Ichiishi, Kalman, Laroque, Mas-Colell, Mitiagin, Rader, Schecter, Smale, and Varian. In particular, the demand functions of the consumers, which we assumed to be fixed, have been treated as variables; economies with many agents, or with many commodities have been taken into account; production has been introduced into the model; demand functions have been replaced as primitive concepts by utility functions or by preference relations, neither of which are restricted to satisfy convexity assumptions. However in most of these extensions the description of the economy requires infinitely many real parameters, i.e., the dimension of the space $\mathcal{E}$ of economies is infinite, and it is no longer possible to use the previous measure theoretical definition of a negligible set. One can now prove only that in the space $\mathcal{E}$, the critical economies form a *closed subset C whose interior is empty*.

The introduction of differential topology into economic theory has made necessary a reexamination of several classical problems, for instance that of differentiable preference relations. In particular, Smale has played a leading role in extending, simplifying, and making more rigorous the study of the set of Pareto optima, and of dynamic processes converging to that set, a program of research in which de Melo, Ong, Simon, Titus, and Y. H. Wan have participated. But of special importance among the applications of Global Analysis to economics has been Smale's recent work on Global Newton Methods where he gives a differential analog of the Scarf-Eaves algorithm for the computation of an economic equilibrium, unifying their approach with the traditional dynamic economic processes of Samuelson, and of Arrow, Block, and Hurwicz.

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