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ECONOMIES WITH A FINITE SET OF EQUILIBRIA[1]

BY GERARD DEBREU

A MATHEMATICAL MODEL which attempts to explain economic equilibrium must have a nonempty set of solutions. One would also wish the solution to be unique. This uniqueness property, however, has been obtained only under strong assumptions,[2] and, as we will emphasize below, economies with multiple equilibria must be allowed for. Such economies still seem to provide a satisfactory explanation of equilibrium as well as a satisfactory foundation for the study of stability provided that all the equilibria of the economy are locally unique. But if the set of equilibria is compact (a common situation), local uniqueness is equivalent to finiteness. One is thus led to investigate conditions under which an economy has a finite set of equilibria.

Now nonpathological examples of economies with infinitely many equilibria can easily be constructed in the case of pure exchange of two commodities between two consumers. Therefore one can at best prove that outside a small subset of the space of economies, every economy has a finite set of equilibria. For the precise definition of "small" in this context, one might think of "null" with respect to an appropriate measure on the space of economies. Such a null set, however, could be dense in the space and a stricter definition is required. Our main result asserts that, under assumptions we will shortly make explicit, outside a null closed subset of the space of economies, every economy has a finite set of equilibria.

The key mathematical tool in the proof is Sard's theorem [**12**, **1**, pp. 37–41, **13**, pp. 45–55], which we now state. Let U be an open subset of R^a and let F be a continuously differentiable function from U to R^b. A point $x \in U$ is *a critical point of* F if the Jacobian matrix of F at x has a rank smaller than b. A point $y \in R^b$ is *a critical value of* F if there is a critical point $x \in U$ with $y = F(x)$. A point of R^b is *a regular value of* F if it is not a critical value.

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[2] An excellent survey of the work done on this uniqueness problem will be found in K. J. Arrow and F. H. Hahn [**2**, Chapter 9].

Sard's theorem: If all the partial derivatives of F to the cth order included, where $c > \max(0, a - b)$, exist and are continuous, then the set of critical values of F has Lebesgue measure zero in R^b.

The economies we consider are pure exchange economies with l commodities and m consumers whose needs and preferences are fixed and whose resources vary. Let L be the set of strictly positive real numbers, P be the set of strictly positive vectors in R^l (i.e., the set of vectors in R^l having all their components strictly positive), and S be the set of vectors in P for which the sum of the components is unity. It is convenient to specify the preferences of the ith consumer by means of his *demand function* f_i, a function from $S \times L$ to $\bar{P}$ such that for every $(p, w_i) \in S \times L$, one has $p \cdot f_i(p, w_i) = w_i$, where the dot denotes inner product in R^l. Given the price vector p in S and his wealth w_i in L, the ith consumer demands the commodity vector $f_i(p, w_i)$ in the closed positive cone $\bar{P}$ of R^l. Having chosen a norm in R^l, we introduce an assumption which will be made for some consumer in the theorem, for every consumer in the proposition:

ASSUMPTION (A): *If the sequence* (p^q, w_i^q) *in* $S \times L$ *converges to* (p^0, w_i^0) *in* $(\bar{S} \setminus S) \times L$, *then* $|f_i(p^q, w_i^q)|$ *converges to* $+\infty$.

Assumption (A) expresses the idea that every commodity is desired by the ith consumer.

An *economy* is defined by $(f_1, \ldots, f_m, \omega_1, \ldots, \omega_m)$, an m-tuple of demand functions, and an m-tuple $\omega = (\omega_1, \ldots, \omega_m)$ of vectors in P. Since the demand functions remain fixed, an economy is actually defined by $\omega \in P^m$.

Given $\omega \in P^m$, an element p of S is an *equilibrium price vector* of the economy ω if

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{m} \omega_i .$$

We denote by $W(\omega)$ the set of p satisfying this equality.

Finally we say that a subset of R^{lm} is *null* if it has Lebesgue measure zero in R^{lm}.

THEOREM: *Given m continuously differentiable demand functions* $(f_1, \ldots, f_m)$, *if some* f_i *satisfies* (A), *then the set of* $\omega \in P^m$ *for which* $W(\omega)$ *is infinite has a null closure.*

PROOF: We assume, without loss of generality, that the first consumer satisfies (A). Let $U = S \times L \times P^{m-1}$, an open set in R^{lm}. We define the function F from U to R^{lm} by associating with the generic element $e = (p, w_1, \omega_2, \ldots, \omega_m)$ of U the value $F(e) = (\omega_1, \omega_2, \ldots, \omega_m)$ where

$$\omega_1 = f_1(p, w_1) + \sum_{i=2}^{m} f_i(p, p \cdot \omega_i) - \sum_{i=2}^{m} \omega_i .$$

We immediately notice that for every $e \in U$, one has $p \cdot \omega_1 = w_1$. We also notice that, given $\omega \in P^m$, the price vector p belongs to $W(\omega)$ if and only if $F(p, p \cdot \omega_1,$

$\omega_2, \ldots, \omega_m) = \omega$ and that the points of $W(\omega)$ are in one-to-one correspondence with the points of $F^{-1}(\omega)$.

Since F is continuously differentiable, by Sard's theorem,

(1) *the set C of critical values of F is null.*

We now want to prove that $C \cap P^m$ is closed relative to P^m. To this end, we first establish that, although the function F may not be proper [**13**, p. 43], it has the closely related property:

(2) *if K is a compact subset of P^m, then $F^{-1}(K)$ is compact.*

Consider a sequence with generic term $e^q = (p^q, w_1^q, \omega_2^q, \ldots, \omega_m^q)$ in $F^{-1}(K)$. We must show that it has a subsequence converging to a point of $F^{-1}(K)$. Let $\omega^q = F(e^q)$ and form the sequence $s^q = (p^q, w_1^q, \omega^q)$. As we noticed earlier, for every $q, p^q \cdot \omega_1^q = w_1^q$. Since ω^q belongs to K, w_1^q is bounded and s^q has a subsequence $\dot{s}^q$ converging to $(p^0, w_1^0, \omega^0) \in \bar{S} \times \bar{L} \times K$. We claim that $p^0 \in S$ and $w_1^0 \in L$. The second relation is a consequence of the equality $p^0 \cdot \omega_1^0 = w_1^0$ obtained in the limit and of the fact that $p^0 \in \bar{S}$ and $\omega_1^0 \in P$. As for the first relation, assume that $p^0 \in \bar{S} \setminus S$. By (A), $|f_1(\dot{p}^q, \dot{w}_1^q)|$ would tend to $+\infty$, a contradiction of the fact that for every q, $f_1(p^q, w_1^q) \leqslant \Sigma_{i=1}^m \omega_i^q$ where the right-hand side is bounded. Summing up, $\dot{e}^q$ converges to $e^0 = (p^0, w_1^0, \omega_2^0, \ldots, \omega_m^0)$ which belongs to U. By continuity of F, one has $F(e^0) = \omega^0$. Therefore $e^0 \in F^{-1}(K)$.

Assertion (2) readily implies

(3) *if E contained in U is closed relative to U, then $F(E) \cap P^m$ is closed relative to P^m.*

Consider a sequence with generic term ω^q in $F(E) \cap P^m$ converging to ω^0 in P^m. The set K consisting of all the ω^q and ω^0 is a compact subset of P^m. For every q, select $e^q \in E$ such that $F(e^q) = \omega^q$. The element e^q belongs to $F^{-1}(K)$ which is compact by (2). Therefore there is a subsequence $\{\dot{e}^q\}$ of $\{e^q\}$ converging to $e^0 \in F^{-1}(K)$. Since every $\dot{e}^q$ belongs to E which is closed relative to U, e^0 belongs to E. By continuity of F, $\omega^0 = F(e^0)$. Thus $\omega^0 \in F(E) \cap P^m$.

As a corollary of (3) we obtain

(4) *$C \cap P^m$ is closed relative to P^m.*

A critical point of F is a point of U at which the Jacobian of F vanishes. Since the Jacobian of F at e is a continuous function of e, the set E of critical points of F is closed relative to U. However, $C = F(E)$.

To complete the proof of the theorem, we remark (see for instance [**10**, p. 8])

(5) *if $\omega \in P^m$ is a regular value of F, then $F^{-1}(\omega)$ is finite.*

Notice first that by (2), $F^{-1}(\omega)$ is compact. Consider an element e of $F^{-1}(\omega)$. At e the Jacobian of F does not vanish. Therefore by the inverse function theorem [**7**, pp. 268–269], there are open neighborhoods U_e of e and V_e of ω homeomorphic under the restriction of F to U_e. In particular e is the only element of $F^{-1}(\omega)$ in U_e. Since $F^{-1}(\omega)$ can be covered by a finite set of open neighborhoods U_e, $F^{-1}(\omega)$ is finite.

In conclusion, if $\omega \in P^m$ is such that $W(\omega)$ is infinite, then $\omega \in C$. By (1), $C \cap P^m$ is null. By (4), so is its closure. *Q.E.D.*

We add a remark to the preceding proof which describes how the set $W(\omega)$ depends on ω.

REMARK: *Under the assumptions of the theorem, if $\omega^0 \in P^m$ is a regular value of F, there are an open neighborhood V of ω^0 and k continuously differentiable functions $g_1, \ldots, g_k$ from V to S such that for every ω in V, the set $W(\omega)$ consists of the k distinct elements $g_1(\omega), \ldots, g_k(\omega)$.*

PROOF: The proof is a variant of the reasoning of [**10**, p. 8]. Let $e_1, \ldots, e_k$ be the elements of $F^{-1}(\omega^0)$. They have pairwise disjoint open neighborhoods $U_1, \ldots, U_k$ homeomorphic, under the restrictions $\gamma_1, \ldots, \gamma_k$ of F to these neighborhoods, to open neighborhoods $V_1, \ldots, V_k$ of ω^0 contained in P^m. Consider $V = \bigcap_{i=1}^k V_i \setminus F(U \setminus \bigcup_{i=1}^k U_i)$.

Since $U \setminus \bigcup_{i=1}^k U_i$ is closed relative to U, the intersection of its image by F and P^m is closed relative to P^m by (3). Since ω^0 does not belong to that image, V is indeed an open neighborhood of ω^0. Let g'_j be the restriction to V of γ_j^{-1}. For every $\omega \in V$, the set $F^{-1}(\omega)$ consists of the k distinct elements $g'_1(\omega), \ldots, g'_k(\omega)$. It now suffices to define $g_j(\omega)$ as $\text{proj}_S\, g'_j(\omega)$ and to recall that the elements of $W(\omega)$ are in one-to-one correspondence with the elements of $F^{-1}(\omega)$. *Q.E.D.*

In particular if $\omega \in P^m$ is a regular value of F and $r(\omega)$ denotes the number of elements of $W(\omega)$, the function r from $P^m \setminus C$ to the set of nonnegative integers is *locally constant*. It is easy therefore to obtain examples of nonempty open sets of economies having multiple equilibria. It suffices to construct an F-regular economy with two commodities and two consumers satisfying the assumptions of the theorem and having several equilibria. There is a neighborhood of that economy in which every economy has the same number of equilibria.

The theorem asserts that excluding a null closed set, every economy in P^m has a finite set of equilibria. Since this set might be empty, this result must be supplemented by the following proposition.

PROPOSITION: *Given m continuous demand functions $(f_1, \ldots, f_m)$, if every f_i satisfies* (A), *then for every $\omega \in P^m$, the set $W(\omega)$ is not empty.*

Since this proposition does not seem to be covered by any of the theorems about the existence of an equilibrium, we give a short proof.

PROOF: Given $\omega \in P^m$, choose a real number t greater than the sum of the components of $\Sigma_{i=1}^m \omega_i$. Let T be the set of vectors x in $\bar{P}$ for which the sum of the components of x is smaller than or equal to t, and let $\hat{T}$ be the set of the vectors x in $\bar{P}$ for which the sum of the components of x equals t. For $i = 1, \ldots, m$ we define the correspondence ϕ_i from $\bar{S} \times L$ to T as follows: if $(p, w_i) \in S \times L$ and $f_i(p, w_i) \in T$, then $\phi_i(p, w_i) = \{f_i(p, w_i)\}$; if $(p, w_i) \in S \times L$ and $f_i(p, w_i) \notin T$, then $\phi_i(p, w_i) = \{x\}$ where x is the intersection point of $\hat{T}$ and the segment $[0, f_i(p, w_i)]$; if $(p, w_i) \in (\bar{S} \setminus S) \times L$, then $\phi_i(p, w_i) = \{x \in \hat{T} | p \cdot x \leqslant w_i\}$.

Next we check that the correspondence ϕ_i is upper hemicontinuous in $\bar{S} \times L$. Consider a sequence $(p^q, w_i^q) \in \bar{S} \times L$ converging to $(p^0, w_i^0) \in \bar{S} \times L$ and a sequence x^q in T converging to x^0 and such that for every q, $x^q \in \phi_i(p^q, w_i^q)$. We wish to show that $x^0 \in \phi_i(p^0, w_i^0)$. If $p^0 \in S$, the continuity of f_i clearly implies $x^0 \in \phi_i(p^0, w_i^0)$. If $p^0 \in \bar{S} \setminus S$, then *either* for infinitely many q, $p^q \in \bar{S} \setminus S$, in which case $[x^q \in \hat{T}$ and $p^q \cdot x^q \leqslant w_i^q]$, *or* for infinitely many q, $p^q \in S$, which implies by (A) that for infinitely many q, $[p^q \in S$ and $f_i(p^q, w_i^q) \notin T]$, in which case also $[x^q \in \hat{T}$ and $p^q \cdot x^q \leqslant w_i^q]$. Therefore in the limit, $x^0 \in \hat{T}$ and $p^0 \cdot x^0 \leqslant w_i^0$.

Now for every $p \in \bar{S}$ define $\psi_i(p) = \phi_i(p, p \cdot \omega_i) - \{\omega_i\}$ and $\psi(p) = \Sigma_{i=1}^m \psi_i(p)$. The correspondence ψ is defined on $\bar{S}$ and takes its values in a compact subset of R^l; it is upper hemicontinuous; for every $p \in \bar{S}$, $\psi(p)$ is convex; for every $p \in \bar{S}$ and every $z \in \psi(p)$, one has $p \cdot z \leqslant 0$. Therefore by [**8**, **11**, **4**, or **5**, 5.6] there are $p^* \in \bar{S}$ and $z^* \leqslant 0$ such that $z^* \in \psi(p^*)$. Consequently for every i, there is $x_i^* \in \phi_i(p^*, p^* \cdot \omega_i)$ such that $\Sigma_{i=1}^m x_i^* \leqslant \Sigma_{i=1}^m \omega_i$. This inequality excludes that for some i, $x_i^* \in \hat{T}$. Thus $p^* \in S$ and for every i, $x_i^* = f_i(p^*, p^* \cdot \omega_i)$. This inequality in turn implies $p^* \cdot x_i^* = p^* \cdot \omega_i$ and, since $p^* \in S$, $\Sigma_{i=1}^m x_i^* = \Sigma_{i=1}^m \omega_i$. *Q.E.D.*

We have assumed that the consumption set X_i of the ith consumer is $\bar{P}$ only to keep the exposition as simple as possible. For instance, the theorem and the proposition extend in a straightforward manner to the case in which X_i is a nonempty, closed, convex subset of R^l bounded below and containing the translate of $\bar{P}$ to each one of the points of X_i.

Differentiability of the demand function has been related to differentiability properties of the utility function in recent studies by Katzner [**9**], Dhrymes [**6**], and Barten, Kloek, and Lempers [**3**]. In particular Katzner makes the following assumptions on the utility function u from $\bar{P}$ to the real line: (i) u is continuous in $\bar{P}$ and twice continuously differentiable in P; (ii) the derivative Du is strictly positive in P; (iii) $[x' \in P,\ x'' \in P,\ x' \neq x'',\ u(x') = u(x'')$, and $0 < t < 1]$ implies $[u(tx'' + (1 - t)x') > u(x')]$; (iv) $u(x) > u(0)$ implies $x \in P$.

Under these assumptions, given $p \in S$ and $w \in L$, there is a unique maximizer $f(p, w)$ of u in the set $\{x \in \bar{P} \mid p \cdot x \leqslant w\}$. The function f is a homeomorphism of $S \times L$ onto P. Its inverse g transforms $x \in P$ into $(p, w) \in S \times L$ where p is the point where the ray from the origin determined by $Du(x)$ intersects S, and $w = p \cdot x$. Katzner observes that the demand function f is continuously differentiable on a dense, open subset of $S \times L$. By another application of Sard's theorem one can actually prove the stronger result—*f is continuously differentiable outside a subset of $S \times L$ which is closed relative to $S \times L$ and has Lebesgue measure zero in R^l.*

Since g is continuously differentiable, the set Γ of its critical values has Lebesgue measure zero in R^l. The set of its critical points is clearly closed relative to P. Therefore its image Γ by the homeomorphism g is closed relative to $S \times L$. At every point of $S \times L \setminus \Gamma$, f is continuously differentiable by the inverse function theorem.

University of California, Berkeley

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