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CONTINUITY PROPERTIES OF PARETIAN UTILITY*

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1. INTRODUCTION

IN THE STUDY of real-valued representations (Paretian utility functions) of a completely preordered space (space of actions preordered by preferences), the continuity properties of the representation can be made to rest on the following result.

$\bar{R}$ will denote the *extended real line*, i.e., the real line with $-\infty$ and $+\infty$ adjoined. A *degenerate* set is a set having at most one element. Given a subset S of $\bar{R}$, a *lacuna* of S is a nondegenerate interval of $\bar{R}$ without points of S but having a lower bound and an upper bound in S; a *gap* of S is a maximal lacuna of S.

THEOREM. *If S is a subset of $\bar{R}$, there is an increasing function g from S to $\bar{R}$ such that all the gaps of $g(S)$ are open.*

The proof will be given in Section 2.

When I introduced this result in the proof of Lemma I of [1], I also asserted that g could be given a certain specific form. In Section 4, I shall prove that this assertion is not correct, using a modification of the triadic set of Cantor due to David A. Freedman. This error does not affect the applications that were made of the result, for in these the specific form of the function g is irrelevant.

In section 3 I shall discuss the use of the theorem in the proof of the continuity properties of Paretian utility functions and, in particular, the recent contribution of John T. Rader [5].

2. PROOF OF THE THEOREM

An *extremal* element of a preordered set is a greatest or a least element of that set.

An *f-set* is a nondegenerate subset A of S such that $(a \in A, b \in A, a < b) \Rightarrow (\{c \in S \mid a \leqq c \leqq b\}$ is a *finite* subset of $A)$.

Consequently, an f-set necessarily has one of the four forms; finite; $a_0 < a_1 < a_2 < \cdots$; $\cdots < a_{-2} < a_{-1} < a_0$; $\cdots < a_{-1} < a_0 < a_1 < \cdots$.

An *i-set* is a nondegenerate subset A of S without extremal points such that $(a \in A, b \in A, a < b) \Rightarrow (\{c \in S \mid a \leqq c \leqq b\}$ is an *infinite* subset

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of A).

A *singular* point of S is a point that belongs to no f-set and to no i-set.

(1) *If A, B are two f-sets (resp. i-sets) and $A \cap B \neq \emptyset$, then $A \cup B$ is an f-set (resp. i-set).*

$A \cup B$ is nondegenerate and has no extremal point if neither A nor B does. It is, therefore, sufficient to check that if $a \in A \cup B$, $b \in A \cup B$ and $a < b$, then $\{c \in S \mid a \leqq c \leqq b\}$ is a finite (resp. infinite) subset of $A \cup B$.

Select c in $A \cap B$ and consider a in A and b in B. If $c \leqq a < b$, then $a \in B$ and the proposition is proved. Similarly, the proposition holds if $a < b \leqq c$. Finally, if $a < c < b$, the proposition follows readily too.

(2) *If C is an f-set (resp. i-set), then there is a maximal f-set (resp. i-set) containing C.*

Take the union of all the f-sets (resp. i-sets) containing C and apply (1).

Define an *F-set* (resp. *I-set*) as a maximal f-set (resp. i-set). Define also an *E-set* as an F-set or an I-set.

(3) *Two distinct E-sets are disjoint.*

If they are both F-sets, or both I-sets, the assertion is obvious. If A is an F-set and B is an I-set, they have at most one point in common. Assume that c is such a common point. A has at least another point a. Now B cannot have points on the same side of c as a. Therefore c is an extremal point of B, a contradiction.

For two subsets A, B of S, the relation $A < B$ means that $(a \in A, b \in B) \Rightarrow (a < b)$.

(4) *If A and B are two distinct E-sets, then $A < B$ or $B < A$.*

Assume that for $a \in A$ and $b \in B$ one has $a < b$ and that for $a' \in A$ and $b' \in B$ one has $b' < a'$. Since A and B are disjoint, two cases may occur: (1) $b' < a$, which with $a < b$ implies $a \in B$, a contradiction; or (2) $a < b'$, which with $b' < a'$ implies $b' \in A$, a contradiction.

(5) *If A is an E-set and a is a singular point, then $\{a\} < A$ or $A < \{a\}$.*

Assume that for $b \in A$ and $c \in A$ one has $b < a < c$; then $a \in A$, a contradiction.

(6) *If S is nondegenerate, then it contains an E-set.*

If there is a pair (a, b) of points in S such that $a < b$ and

$\{c \in S \mid a \leqq c \leqq b\}$ is finite, then the latter set is an f-set. By (2) there is an F-set containing it.

If for every pair (a, b) of points in S such that $a < b$ the set $\{c \in S \mid a \leqq c \leqq b\}$ is infinite, then, by removing from S its extremal points if any, one obtains an I-set. Q.E.D.

Therefore, a nondegenerate S can be partitioned into its E-sets and the (possibly empty) set of its singular points. The case of a degenerate S, for which the theorem is trivially true, will be excluded until the end of this section.

According to (4), if A, B are two distinct E-sets, the open intervals $]\text{Inf } A, \text{Sup } A[$ and $]\text{Inf } B, \text{Sup } B[$ are disjoint. Since the intervals associated with the E-sets in this fashion are nondegenerate, their class is countable (possibly finite), and so is theclass of the E-sets. The latter can therefore be arranged in a sequence $E_1, E_2, \cdots$, all of whose elements are understood to be distinct.

(7) *If E_n is an F-set and α_n, β_n are two points of $\bar{R}$ such that $\alpha_n < \beta_n$, then there is an increasing function g_n from E_n to $\bar{R}$ such that all the gaps of $g_n(E_n)$ are open, $\alpha_n = \text{Inf } g_n(E_n)$ and $\beta_n = \text{Sup } g_n(E_n)$.*

This is obvious if one recalls the four possible forms of an F-set.

(8) *If E_n is an I-set and α_n, β_n are two points of $\bar{R}$ such that $\alpha_n < \beta_n$, then there is an increasing function g_n from E_n to $\bar{R}$ such that $g_n(E_n)$ has no gap, $\alpha_n = \text{Inf } g_n(E_n)$ and $\beta_n = \text{Sup } g_n(E_n)$.*

Let D_n be a countable dense subset of E_n. Following the procedure of [2, (Section 4.6)], one can first define an increasing function from D_n *onto* the rationals of the open interval $]\alpha_n, \beta_n[$. Then by extending that function from D_n to E_n, one obtains an increasing function g_n from E_n to the open interval $]\alpha_n, \beta_n[$. Since all the rationals of this interval belong to $g_n(E_n)$, the latter set has no gap. Q.E.D.

The function g from S to $\bar{R}$ is now defined inductively. To this effect, we take for each E_n first a pair (α_n, β_n) of points of $\bar{R}$ such that $\alpha_n < \beta_n$ and then a function g_n satisfying the conditions of (7) if E_n is an F-set, of (8) if E_n is an I-set. Two E-sets are said to be *adjacent* if there is at most one point of S between them.

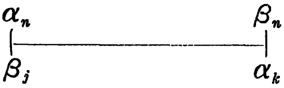
For E_1, take $\alpha_1 = 0, \beta_1 = 1$.

For E_n, consider the three cases that may arise.

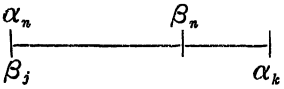
(1) $E_m < E_n$ for every $m < n$. Let E_j be the greatest of these E_m. If E_j and E_n are adjacent, take $\alpha_n = \beta_j$ and $\beta_n = \beta_j + 1$. If E_j and E_n are not adjacent, take $\alpha_n = \beta_j + 1$ and $\beta_n = \beta_j + 2$.

(2) $E_m > E_n$ for every $m < n$. In this case replace in (1) j by k, "greatest" by "least", α by β, β by α and $+$ by $-$.

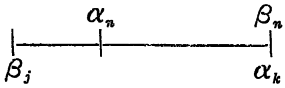
(3) There are some $E_m < E_n$ with $m < n$; let E_j be the greatest of these. There are also some $E_m > E_n$ with $m < n$; let E_k be the least of these.



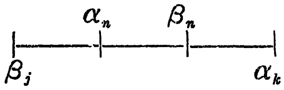
If E_j, E_n are adjacent and E_n, E_k are too, take $\alpha_n = \beta_j$ and $\beta_n = \alpha_k$.



If E_j, E_n are adjacent but E_n, E_k are not, take $\alpha_n = \beta_j$ and $\beta_n = (\beta_j + 2\alpha_k)/3$.



If E_n, E_k are adjacent but E_j, E_n are not, take $\alpha_n = (2\beta_j + \alpha_k)/3$ and $\beta_n = \alpha_k$.



If neither E_j, E_n nor E_n, E_k are adjacent, take $\alpha_n = (2\beta_j + \alpha_k)/3$ and $\beta_n = (\beta_j + 2\alpha_k)/3$.

g coincides with g_n on E_n for every n. It remains to define g for the singular points of S. Let a be one of them. If $\{a\} < E_n$ for every n, define $g(a)$ as $\operatorname{Inf} \alpha_n$ (which may be $-\infty$). If $E_n < \{a\}$ for every n, define $g(a)$ as $\operatorname{Sup} \beta_n$(which may be $+\infty$). To dispose of the residual case for which there are E_m and E_n such that $E_m < \{a\} < E_n$, we prove

(9) *If the class of the E-sets is partitioned into two nonempty classes $\mathscr{A}$ and $\mathscr{B}$ such that $(E_r \in \mathscr{A}$ and $E_s \in \mathscr{B}) \Rightarrow (E_r < E_s)$, then* $\operatorname{Sup}_{E_r \in \mathscr{A}} \beta_r = \operatorname{Inf}_{E_s \in \mathscr{B}} \alpha_s$.

Given n large enough, let E_{p_n} be the greatest of the E_m in $\mathscr{A}$ such that $m < n$, and let E_{q_n} be the least of the E_m in $\mathscr{B}$ such that $m < n$. If, for some n, E_{p_n} and E_{q_n} are adjacent, then $\beta_{p_n} = \alpha_{q_n}$ and the proposition is proved. If E_{p_n} and E_{q_n} are adjacent for no n, then, for every n, the interval $[\beta_{p_{n+1}}, \alpha_{q_{n+1}}]$, which is contained in the interval $[\beta_{p_n}, \alpha_{q_n}]$, either has the same length or has a length three times smaller. Since the latter must occur for infinitely many n, the proposition is also proved in this situation. Q.E.D.

Going back to the residual case decribed before the statement of (9), we define $\mathscr{A}$ as the class of all the $E_r < \{a\}$ and $\mathscr{B}$ as the class of all the $E_s > \{a\}$. These two classes satisfy the conditions of (9). Therefore $\operatorname{Sup}_{E_r < \{a\}} \beta_r = \operatorname{Inf}_{\{a\} < E_s} \alpha_s$. By definition, $g(a)$ is this common value.

To check that the function g from S to $\bar{R}$ is increasing, consider a and b in S such that $a < b$. If a and b belong to the same E_n, obviously

$g(a) < g(b)$. If $a \in E_m$, $b \in E_n$ and E_m, E_n are not adjacent, then it is again clear that $g(a) < g(b)$. If $a \in E_m$, $b \in E_n$ and E_m, E_n are adjacent, then $\beta_m = \alpha_n$; one cannot have $g_m(a) = \beta_m$ and $g_n(b) = \alpha_n$, otherwise E_m would be an F-set finite on the right and E_n would be an F-set finite on the left; consequently, neither E_m nor E_n would be a maximal f-set; thus, $g(a) < g(b)$ in this case too. If a is singular, $b \in E_n$, and there is an E_m between $\{a\}$ and E_n, then clearly $g(a) < g(b)$. If a is singular, $b \in E_n$, and there is no E_m between $\{a\}$ and E_n, then $g(a) = \alpha_n$; one cannot have $g_n(b) = \alpha_n$, otherwise E_n would be an F-set finite on the left; consequently a would not be singular and E_n would not be a maximal f-set; thus, $g(a) < g(b)$. If b is singular and $a \in E_n$, the last two sentences are altered in an obvious way. Finally, if a and b are singular, there must be an E_n between them; therefore, $g(a) < g(b)$.

The last step in the proof consists in checking that all the gaps of $g(S)$ are open. Let then G be a gap of $g(S)$. We will prove by contradiction that G is a gap of some $g(E_n)$. Assume that such is not the case, and let $\mathscr{A}$ be the class of all the E_r such that $\beta_r \leqq \operatorname{Inf} G$, let $\mathscr{B}$ be the class of all the E_s such that $\operatorname{Sup} G \leqq \alpha_s$. Since $\mathscr{A}$ and $\mathscr{B}$ exhaust the class of the E-sets, and since $\operatorname{Inf} G < \operatorname{Sup} G$, it follows from (9) that either $\mathscr{A}$ or $\mathscr{B}$ is empty. If $\mathscr{A}$ is empty, there is exactly one point of $g(S)$ that is $\leqq \operatorname{Inf} G$. This point must be $\operatorname{Inf} G$ and be the image of a singular point a. Thus $g(a) < \operatorname{Sup} G$, while $\operatorname{Sup} G \leqq \operatorname{Inf}_{E_s \in \mathscr{B}} \alpha_s$. As $\mathscr{B}$ is the class of all the E_n, the inequality $g(a) < \operatorname{Inf}_{E_s \in \mathscr{B}} \alpha_s$ contradicts the definition of $g(a)$. If $\mathscr{B}$ is empty, one obtains a contradiction in the same way. Therefore, G is a gap of some $g(E_n)$. However, if E_n is an I-set, $g(E_n)$ has no gap; if E_n is an F-set, all the gaps of $g(E_n)$ are open.

3. CONTINUITY PROPERTIES OF THE REPRESENTATION

The reasoning of Lemma I of [1] can now be repeated. Consider a *preordered* set X, i.e., a set X and a reflexive, transitive binary relation on X denoted $\precsim$. For two elements y and z of X, we define "$y \prec z$" as "$y \precsim z$ and not $z \precsim y$". For two subsets Y and Z of X, we define "$Y \prec Z$" as "$y \prec z$ for every y in Y and z in Z". The *upper* (resp. *lower*) *preorder topology* on X is the weakest topology for which the set $\{x \in X \mid x \succsim y\}$ (resp. $\{x \in X \mid x \precsim y\}$) is closed for every y in X. The set X is said to be *completely preordered* by $\precsim$ if $[y \in X \text{ and } z \in X] \Rightarrow [y \precsim z \text{ and/or } z \precsim y]$.

COROLLARY. *Let X be a completely preordered set. If there is an increasing function v from X to $\bar{R}$, then there is an increasing function u from X to $\bar{R}$ that is upper semicontinuous in the upper preorder* *topology and lower semicontinuous in the lower preorder topology.*

PROOF. Apply the theorem to the subset $v(X)$ of $\bar{R}$. There is an increasing function g from $v(X)$ to $\bar{R}$ such that all the gaps of $g(v(X))$ are open. Define u by $u(x) = g(v(x))$. Thus u is an increasing function from X to $\bar{R}$ and all the gaps of $u(X)$ are open.

We will only consider the upper preorder topology on X and establish the upper semicontinuity of u by proving that, for every γ in $\bar{R}$, the inverse image $u^{-1}([\gamma, +\infty])$ is closed. A similar proof can be given for lower semicontinuity in the lower preorder topology.

If $\gamma \in u(X)$, i.e., if there is y in X such that $u(y) = \gamma$, then $u^{-1}([\gamma, +\infty]) = \{x \in X \mid x \gtrsim y\}$, a closed set.

If $\gamma \notin u(X)$ and γ is not in a gap of $u(X)$, either $\gamma \leqq \operatorname{Inf} u(X)$, in which case $u^{-1}([\gamma, +\infty]) = X$, a closed set; or $\operatorname{Sup} u(X) \leqq \gamma$, in which case $u^{-1}([\gamma, +\infty]) = \emptyset$, a closed set; or $[\gamma, +\infty] = \bigcap_{\alpha \in u(X), \alpha \leqq \gamma} [\alpha, +\infty]$, in which case $u^{-1}([\gamma, +\infty]) = \bigcap_{\alpha \in u(X), \alpha \leqq \gamma} u^{-1}([\alpha, +\infty])$, a closed set since it is an intersection of closed sets.

If γ is in a gap of $u(X)$, this gap is an open interval $]\lambda, \mu[$, where λ and μ belong to $u(X)$. Therefore $u^{-1}([\gamma, +\infty]) = u^{-1}([\mu, +\infty])$, a closed set. Q.E.D.

PROPOSITION 1 (Rader [5]). *Let X be a completely preordered topological space having a countable base of open sets. If the set $\{x \in X \mid x \gtrsim y\}$ is closed for every y in X, then there is an upper semicontinuous, increasing function u from X to $\bar{R}$.*

PROOF. We first construct an increasing function v from X to $\bar{R}$ following Rader. Let $O_1, O_2, \cdots$ be the open sets in the countable base. Given an element x of X, define

$$N(x) = \{n \mid O_n \prec \{x\}\} \quad \text{and} \quad v(x) = \sum_{n \in N(x)} \frac{1}{2^n},$$

adopting the convention that a sum over an empty set of indices is 0.

If $y \precsim z$, then $N(y)$ is a subset of $N(z)$ and $v(y) \leqq v(z)$. Thus v is nondecreasing. It is actually increasing, for if $y \prec z$, then y belongs to the open set $\{x \in X \mid x \prec z\}$. Consequently, for some n, one has $y \in O_n \prec \{z\}$. For that n, one does not have $O_n \prec \{y\}$; therefore $N(y)$ is a proper subset of $N(z)$ and $v(y) < v(z)$.

We now apply the corollary and obtain an increasing function u from X to $\bar{R}$ upper semicontinuous in the *upper preorder topology*. Since the given topology on X is stronger than or equal to the upper preorder topology, the proof is completed.

PROPOSITION 2. *Derived from Proposition 1 by substituting $\precsim$ for*

$\gtrsim$ *and "lower" for "upper" does not require a new proof.*

Theorem II of [1] states

PROPOSITION 3. *Let X be a completely preordered topological space having a countable base of open sets. If the sets $\{x \in X \mid x \gtrsim y\}$ and $\{x \in X \mid x \lesssim y\}$ are closed for every y in X, then there is a continuous, increasing function u from X to $\bar{R}$.*

PROOF. According to Proposition 1, there is an increasing function v from X to $\bar{R}$. Therefore, according to the Corollary, there is an increasing function u from X to $\bar{R}$ upper semicontinuous in the upper preorder topology and lower semicontinuous in the lower preorder topology. Since the given topology on X is stronger than or equal to both these topologies, u is both upper and lower semicontinuous in the given topology. Q.E.D.

As another consequence of the corollary we prove

PROPOSITION 4 (Eilenberg [3]). *Let X be a completely preordered, connected, separable topological space. If the sets $\{x \in X \mid x \gtrsim y\}$ and $\{x \in X \mid x \lesssim y\}$ are closed for every y in X, then there is a continuous, increasing function u from X to $\bar{R}$.*

PROOF. Let $Z = \{z_1, z_2, \cdots\}$ be a countable dense subset of X. Consider y and y' in X such that $y \prec y'$. The two sets $\{x \in X \mid x \lesssim y\}$ and $\{x \in X \mid y' \lesssim x\}$ are closed, nonempty, and disjoint. Therefore, they do not exhaust X, which is connected. Since the open set $\{x \in X \mid y \prec x \prec y'\}$ is not empty, it owns some z_n in Z. We now construct an increasing function v from X to $\bar{R}$, following Milgram [4, (26–27)]. Given an element x of X, define

$$N(x) = \{n \mid z_n \prec x\} \quad \text{and} \quad v(x) = \sum_{n \in N(x)} \frac{1}{2^n}.$$

If $y \lesssim y'$, then $N(y)$ is a subset of $N(y')$ and $v(y) \leqq v(y')$. Thus v is nondecreasing. It is actually increasing, for if $y \prec y'$, then for some n, one has $y \prec z_n \prec y'$. Therefore $N(y)$ is a proper subset of $N(y')$ and $v(y) < v(y')$.

Applying the Corollary, one obtains, as in the proof of Proposition 3, a continuous increasing function u from X to $\bar{R}$. Q.E.D.

Finally, we give a proposition that, in conjunction with the Corollary, slightly strengthens Lemma II of [1].

PROPOSITION 5. *Let X be a completely preordered set having a countable subset Z such that if x and y in X satisfy $x \prec y$, then there is z in Z satisfying $x \lesssim z \lesssim y$. Then there is an increasing* *function v from X to $\bar{R}$.*

PROOF. Let A be the set of the indifference classes of X, and let C be the set of the indifference classes of X having an element in Z. The set A is completely ordered by the relation $\leqq$ induced by $\precsim$, and it has a countable subset C playing for A the role that Z played for X. We now consider the pairs (b, b') of elements of A such that $b \prec b'$ and there is no a in A satisfying $b \prec a \prec b'$. Thus, b and/or b' must be in C. Consequently, these pairs form a countable class, since at most two of them have a given element of C as a component. We add the components of the pairs in that countable class to C and obtain a new countable set $C^* = \{c_1, c_2, \cdots\}$. Given an element x of X, we denote the indifference class of x by $a(x)$ and define, as in the proof of Proposition 4,

$$N(x) = \{n \mid c_n \prec a(x)\} \quad \text{and} \quad v(x) = \sum_{n \in N(x)} \frac{1}{2^n}.$$

4. AN EXAMPLE

In this section, S denotes a bounded subset of the real line. Given a point x of S, let $f(x)$ be the sum of the lengths of the non-open gaps G of S such that $\operatorname{Sup} G \leqq x$. The function g of the theorem could be given the specific form $g(x) = x - f(x)$ only if $[x \in S, x' \in S, x < x']$ implied $[f(x') - f(x) < x' - x]$. To prove that g cannot be so restricted it suffices to exhibit a subset S of $[0, 1]$ such that 0 and 1 belong to S, S has no open gap, and the sum of the lengths of the gaps of S is one. In other words, S should be such that $S \subset [0, 1]$; $0 \in S$; $1 \in S$; $[a \in S, b \in S, a < b] \Rightarrow$ [there is c in S satisfying $a < c < b$]; the measure of the closure of S is zero. The following set having all these properties has been communicated to me by David A. Freedman. The ternary number system is used. S is made up of 0, 1 and all the reals between 0 and 1 having ternary expansions $0. p_1 p_2 \cdots$ where (i) for all n, $p_n \neq 1$, (ii) for infinitely many n, $p_n = 0$, (iii) for infinitely many n, $p_n = 2$. It is clear that $[a \in S, b \in S, a < b] \Rightarrow$ [there is c in S satisfying $a < c < b$] and that S is contained in the triadic set C of Cantor which is made up of all the reals in $[0, 1]$ whose ternary expansions satisfy (i). However C is closed and has measure zero.

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